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LARGE WAVE NUMBER APERTURE LIMITED FOURIER INVERSION AND INVERSE SCATTERING

by

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ABSTRACT

Aperture limited Fourier operators are like pseudo-differential operators, except that the symbol of the latter is part of the unknown kernel function of the former. Such operators arise naturally in inverse scattering when one applies an integral inverse scattering operator to model data with an unknown (such as a reflection coefficient) which depends on the source/receiver location as well on interior medium coordinates. Large wave number aperture limited Fourier operators can be analyzed by multi-dimensional stationary phase. The nature of the asymptotic expansion of such operators will then depend on the properties of the kernel function to which the operator is applied. Of interest in inverse scattering are distributions with support at a point or on a surface (the singular function of a surface) and piecewise smooth functions (whose discontinuity surfaces are reflectors).

The purpose of this paper is to present some features of the asymptotics of large wave number aperture limited Fourier inversion operators and to relate these results to the application of a recently developed inverse scattering formalism as applied to Kirchhoff-approximate scattering data from a reflecting surface. It is shown how the latter problem can be reduced to the former, asymptotically. Then, the more simply derived asymptotic expansions of the former can be applied to predict the output of the latter.

TABLE OF CONTENTS

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INTRODUCTION

This is another in a series of papers [Bleistein, 1987a,b] which analyzes by classical methods an important advance in inverse scattering proposed by Beylkin [1985]. The method applies to a variety of source/receiver configurations in a fairly arbitrary host medium with the viability of the experimental arrangement characterized by the nonvanishing of a certain Jacobi determinant. Beylkin confirms the validity of his technique through the application of pseudo-differential operator theory, which predicts the validity of the result in a large wave number limit. Thus, those features of the unknown function which dominate the large wave number Fourier transform of the function, will dominate the output when his inversion is applied to model data. For example, when the unknown is piecewise smooth, the discontinuities of the unknown function (reflectors) will dominate, followed by discontinuities of the first derivative, etc.

Bleistein [1987a], proposes a modification of the Beylkin inversion operator designed to process data from piecewise smooth functions and produce a reflector map. That is, at each discontinuity of the medium parameters, this operator produces a band-limited delta function which peaks on the discontinuity surface — *the singular function of the surface* — scaled by a factor from which the change in medium parameters across the surface can be estimated. That paper deals with the case in which only the propagation speed varies and Bleistein [1987b] addresses the problem in which both propagation speed and density vary across the reflector. Current research focuses on isotropic and anisotropic elastic wave field inversion. The inversion operator takes the form of an integration over the data collection surface. It has the structure of a Kirchhoff migration operator [Schneider, 1978] and, therefore, is called *Kirchhoff inversion.*

In both of the above cited papers by this author, the validity of the inversion operator was confirmed by applying the operator to Kirchhoff-approximate data from a single reflector and analyzing the output by multi-dimensional stationary phase applied simultaneously to the integration over the data collection surface and the reflecting surface. Stationarity conditions from the latter integral merely impose Snell's law on the ray trajectories from a source and receiver to the reflecting surface. Stationarity conditions on the former integral ties the integration variables to the output point in a manner that is dependent on the source/receiver configuration.

The main point of this paper is that the analysis of this multi-fold integral can be reduced by classical methods to the analysis of large wave number aperture limited Fourier transform-like integrals of functions which depend on both the spatial and wave vector variables.

This is exactly the structure of a pseudo-differential operator. However, in application to inverse scattering (in particular, when the modeling data is Kirchhoff data), the symbol of the pseudo-differential operator is part of the unknown function. Thus, important results from that theory do not seem to apply here. Furthermore, the classical analysis of these integrals leads to detailed information about the structure of the output of this type of inversion for different classes of functions of interest in inverse scattering. In particular, the relationship between the peak amplitude of the output and reflectivity is accessible in the classical theory and has not yet been deomonstrated in the pseudo-differential operator theory.

Aperture limited Fourier transforms have the advantage that the phase of the integrand is linear in both the spatial and Fourier variables, making the application of the method of stationary phase much easier than it is for the analysis of the inverse scattering operator. In the latter, the phase is a sum of differences of travel time trajectories. The sum is from source and receiver to either reflection surface point or output point. The trajectories are the rays of geometrical optics characterized in the general case only by the differential equations which they satisfy.

The following section presents the asymptotic analysis of Fourier integral operators on piecewise smooth functions of the spatial variable, but with the additional feature that they depend on the unit vector in the dual transform variable, as well.

This is followed by some numerical examples which demonstrate the theory.

The fourth section introduces the application of Kirchhoff inversion to Kirchhoff data from a single reflector. It is shown there that consideration of output points anywhere but in a neighborhood of the reflector can be neglected under reasonable assumptions and then that the integration over the reflector need only be carried out "nearby" this output point. In this small region, there exists a transformation of coordinates from frequency and spatial variables over the observation surface to wave vector variables. This transformation has the property that it transforms the multi-traveltime phase function of the integral operator into the Fourier dot product of wave vector with spatial variables. At that point, the integral is reduced to one of the type considered in the previous sections and the asymptotic output is known from the results of those sections.

Thus, this type of travel time inversion is seen to be a generalized forward and inverse Fourier transform of a function of both spatial and wave vector variables. This is suggested in Beylkin [1985], although that author seems to prefer a space-time formulation of the solution and interpretation of the result as a generalized Radon transform.

APERTURE LIMITED FOURIER IDENTITY OPERATORS

Consider the integral

$$I = \frac{1}{(2\pi)^3} \int_{D_k} d^3k \int_{D_{x'}} d^3x' \, f\left(\boldsymbol{x}'/L, \hat{\boldsymbol{k}}\right) \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')\right] \quad . \tag{1}$$

Here, $\boldsymbol{x}$, $\boldsymbol{x}'$ and $\boldsymbol{k}$ are three component vectors and $\hat{\boldsymbol{k}}$ is a unit vector. If the domain D_k

were infinite in extent, and the function f were independent of $\hat{\boldsymbol{k}}$, then the integral, I, would just be equal to $f(\boldsymbol{x}/L)$ for $\boldsymbol{x}$ in $D_{x'}$ and zero, outside, for a broad class of functions f. This is just Fourier inversion. The structure of the analysis, then, comes from the nature of the domain, D_k, and the additional dependence of f on $\hat{\boldsymbol{k}}$.

The function f is assumed to have as many derivatives as necessary to carry out all the differentiations below. For functions that are only piecewise smooth enough, decompose the domain of integration into separate domains whose boundaries include all of the discontinuities of f in $\boldsymbol{x}'$. Then, the integral over each subdomain is of the type defined here. It will be seen below that, asymptotically, the integral depends primarily on the boundary values of the integrand at certain critical points. The manner in which the critical points are determined will make it clear that for a sum of such integrals, the output will depend on the jump in the integrand across these discontinuity surfaces. That is, a critical point on a boundary surface for one integral will simultaneously be a critical point for the other integral sharing the same piece of boundary.

The length scale, L, is assumed to characterize the size of the derivatives of f. That is, derivatives with respect to $\boldsymbol{y}' = \boldsymbol{x}'/L$ should be comparable in size to f, itself. For convenience, the same parameter L is used to characterize the length scales of the bounding surface of $D_{x'}$. For example, L might be a "typical" principal radius of curvature or a lower bound of the principal radii of curvature for the boundary of D_x.

In application to inverse scattering, the domain, D_k, is symmetric with respect to the origin. That is, whenever $\hat{\boldsymbol{k}}$ is in D_k, so is $-\hat{\boldsymbol{k}}$. The reason for this is that $\boldsymbol{k}$ is proportional to a frequency, ω, thus including $-\boldsymbol{k}$ in D_k whenever it includes $+\boldsymbol{k}$. Thus, symmetry of D_k will be assumed below, although it is not essential to the analysis. Furthermore, it is assumed that D_k does not contain the origin. In fact, denote by K the minimum distance from the origin to the domain D_k. Now introduce the dimensionless variables,

$$\boldsymbol{p} = \boldsymbol{k}/K\,, \quad \boldsymbol{y}' = \boldsymbol{x}'/L\,, \quad \lambda = KL \quad , \tag{2}$$

and rewrite (1) as

$$I(\lambda, \boldsymbol{y}) = \left[\frac{\lambda}{2\pi}\right]^3 \int_{D_p} p^2 dp \sin\theta \, d\theta \, d\phi \int_{D_{y'}} d^3y' \; f(\boldsymbol{y}', \hat{\boldsymbol{p}}) \exp\left[i\lambda p \, \hat{\boldsymbol{p}} \cdot (\boldsymbol{y} - \boldsymbol{y}')\right] \quad . \tag{3}$$

In this equation, p, θ, ϕ, are polar coordinates, and the unit vector, $\hat{\boldsymbol{p}}$, is given in terms of θ and ϕ by

$$\hat{\boldsymbol{p}} = (\sin\theta\cos\phi, \, \sin\theta\sin\phi, \, \cos\theta) \quad . \tag{4}$$

The domains, $D_{y'}$ and D_p, are the images of $D_{x'}$ and D_k, respectively, under the scaling (2). In particular, the minimum distance from the origin in $\boldsymbol{p}$ to D_p is equal to unity.

The objective now is to analyze $I(\lambda, \boldsymbol{y})$ asymptotically for large λ as $\boldsymbol{y}$ varies over some domain which includes $D_{y'}$. This is aperture-limited Fourier inversion when the aperture is such that the length scales of f and its support in the x-domain are "many" wavelengths for all of the available information in the k-domain. Formally, this is the asymptotic expansions of $f(\lambda, \boldsymbol{y})$ as $\lambda \rightarrow \infty$.

The product, $\lambda p \geq \lambda >> 1$, appears as a "natural" large parameter in the integral (3). Thus, multi-dimensional stationary phase in the five variables, $\boldsymbol{y}'$, θ, ϕ will be carried out, with λp as a formal large parameter, followed by integration with respect to p.

The Significance of the Boundary Values in $D_{y'}$

Define

$$\Phi = \hat{\boldsymbol{p}} \cdot (\boldsymbol{y} - \boldsymbol{y}') \tag{5}$$

and note that

$$\nabla' \Phi = -\hat{\boldsymbol{p}} \neq \boldsymbol{0} \ . \tag{6}$$

Here, ∇' is the gradient of Φ with respect to the three variables, $\boldsymbol{y}'$. Since this gradient is never zero, there can be no stationary points of the fivefold integral. Thus, proceed to calculate the asymptotic expansion of the integral (3) by using integration by parts (the divergence theorem). To do so, first rewrite the integrand as

$$f(\boldsymbol{y}', \hat{\boldsymbol{p}}) \exp\left[i\lambda p \Phi\right] = \frac{1}{i\lambda p}\left\{\nabla' \cdot \left[\hat{\boldsymbol{p}} f(\boldsymbol{y}', \hat{\boldsymbol{p}}) \exp\left[i\lambda p \Phi\right]\right] + f_1(\boldsymbol{y}', \hat{\boldsymbol{p}}) \exp\left[i\lambda p \Phi\right]\right\},$$

$$f_1(\boldsymbol{y}', \hat{\boldsymbol{p}}) = -\nabla' \cdot \left[\hat{\boldsymbol{p}} f(\boldsymbol{y}', \hat{\boldsymbol{p}})\right] \ . \tag{7}$$

When this is substituted into (3) and the divergence theorem is applied to the first term, the result is

$$I(\lambda, \boldsymbol{y}) = I_0(\lambda, \boldsymbol{y}) + \frac{1}{i\lambda} I_1(\lambda, \boldsymbol{y}) \ , \tag{8}$$

$$I_0(\lambda, \boldsymbol{y}) = \frac{-\lambda^2}{[2\pi]^3} \int_{D_p} p\,dp \sin\theta\,d\theta\,d\phi \cdot \int_{S_{y'}} dS_{y'}\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}\; f(\boldsymbol{y}', \hat{p}) \exp\left[i\lambda p\Phi\right] \;,$$

$$I_1(\lambda, \boldsymbol{y}) = \left[\frac{\lambda}{2\pi}\right]^3 \int_{D_p} p\,dp \sin\theta\,d\theta\,d\phi \int_{D_{y'}} d^3y'\; f_1(\boldsymbol{y}', \hat{\boldsymbol{p}}) \exp\left[i\lambda p\Phi\right] \;. \qquad (9)$$

In this equation, $\hat{\boldsymbol{n}}$ denotes the normal to the boundary, $S_{y'}$, of the domain $D_{y'}$. It should be noted that the integral $I_1(\lambda, \boldsymbol{y})$ is exactly like $I(\lambda, \boldsymbol{y})$, itself. However, in (8), it is multiplied by $1/i\lambda$. Now consider the effect of repeating this process with $I_1(\lambda, \boldsymbol{y})$. We would again obtain an integral like $I_0(\lambda, \boldsymbol{y})$, but multiplied by another power of $1/i\lambda$. Continuing recursively, we obtain an asymptotic series of integrals over D_p and $S_{y'}$. Thus, it is reasonable to conclude that the leading order asymptotic expansion of I must come from the analysis of $I_0(\lambda, \boldsymbol{y})$, unless, of course, $f\,(\boldsymbol{y}', \hat{\boldsymbol{p}})$ were identically zero on $S_{y'}$. In fact, this last observation leads to the following result.

Lemma 1:

Suppose that $f\,(\boldsymbol{y}', \hat{p})$ is infinitely differentiable in $D_{y'}$ and vanishes "infinitely smoothly" on the boundary, $S_{y'}$. Then $I(\lambda, \boldsymbol{y})$ is asymptotically zero to all orders of $1/\lambda$.

Proof:

In (8), $I_0(\lambda, \boldsymbol{y})$ is zero because of the assumptions on f. However, the assumptions on f are true for f_1, as well. Thus, repeat the integration by parts process and obtain another integral like $I(\lambda, \boldsymbol{y})$ but multiplied now by $1/(i\lambda)^2$, with an integrand f_2, which also satisfies the conditions placed on f. Repeat the process recursively and obtain any desired algebraic power of $1/i\lambda$ as a multiplier. This completes the proof.

An Example

As an example of this result, consider

$$f(\boldsymbol{x}') = \begin{cases} 0, & x'_3 \le 0, \\ \sqrt{L/x'_3}\,\exp\left[-L/x'_3\right], & x'_3 > 0 \;. \end{cases} \qquad (17)$$

Note that this function is infinitely differentiable for all $\boldsymbol{x}'$. Assume that $D_{x'}$ is all space and that D_k is the symmetric domain $K < |k_3| < K_1$. After integrating in all of the variables except k_3, I, as defined by (1), is given by

$$I = \frac{L}{\sqrt{\pi}} \int_{-K_1}^{-K} + \int_{K}^{K_1} \frac{dk_3}{\Lambda} \exp\left[-\Lambda e^{i\pi/4sgnk_3} + ik_3 z + i\pi/4sgnk_3 \right] \quad . \tag{11}$$

In this equation,

$$\Lambda = \sqrt{|k_3| L} \quad . \tag{12}$$

This result can be obtained by using results about the modified Bessel function to be found in Watson [1980] and Abramowitz and Stegun, [1965]. From this result, it follows that

$$I = o\left[\exp\left[-\sqrt{2\lambda} \right]\right] , \quad \lambda = KL \quad . \tag{13}$$

This estimate is independent of z. With a little more effort, one can show that, for example, when $\lambda = 3$, $|I| < .04$, independent of z. On the other hand, directly from (10), $f(0,0,L/5) \approx 1.83$. Thus, the aperture-limited Fourier inversion provides a poor approximation of this infinitely differentiable function for "large" values of λ.

Remark:

This example demonstrates the lemma (although the infinite domain of the example requires a little extra effort). In this example, the decay is actually exponential in $\sqrt{\lambda}$. In general, one can only predict "faster than algebraic" decay.

The point of the lemma is that the integrand in (8) only has boundary critical points and then, only if the function $f(\mathbf{y}', \hat{p})$ does not vanish infinitely smoothly. That is, it is the discontinuities of f that dominate the integration. (A boundary point where f does not vanish infinitely smoothly is, after all, a discontinuity of the function or one of its derivatives, since its interior limit is not equal to its exterior limit.) Thus, for this class of functions, the large wave number aperture-limited Fourier integral operator in (1) is not an identity operator, but, at best, it is an operator whose output depends only on the discontinuities of f (or its derivatives) in some as yet undetermined way.

Stationary Phase Analysis for I_0

Consider, now, the asymptotic analysis of $I_0(\lambda, \mathbf{y})$, in (9). Again the phase is given by (5), except that $\mathbf{y}'$ is a function of two surface parameters, say, σ_1 and σ_2. Thus, fourfold stationary phase in the variables, σ_1, σ_2, θ, and ϕ will be applied to this integral. The first derivatives of Φ are given by

$$\frac{\partial\Phi}{\theta\sigma_i} = -\hat{\boldsymbol{p}}\cdot\frac{\partial \boldsymbol{y}'}{\partial\sigma_i}\,,\quad i=1\,,\,2\,,\quad \frac{\partial\Phi}{\partial\theta} = \hat{\boldsymbol{\theta}}\cdot(\boldsymbol{y}-\boldsymbol{y}')\,,\quad \frac{\partial\Phi}{\partial\phi} = \sin\theta\,\hat{\boldsymbol{\phi}}\cdot(\boldsymbol{y}-\boldsymbol{y}')\quad ,$$

$$\hat{\boldsymbol{\theta}} = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)\quad ,$$

$$\hat{\boldsymbol{\phi}} = (-\sin\phi,\ \cos\phi,\ 0)\quad . \tag{14}$$

Note that the vectors $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are orthogonal to $\hat{\boldsymbol{p}}$. Setting these four first derivatives equal to zero has the following geometrical interpretation. At the stationary point, $\hat{\boldsymbol{p}}$ must be orthogonal to two linearly independent tangent vectors in $S_{y'}$. Thus, $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{n}}$ must be colinear or anti-colinear. Furthermore, $\boldsymbol{y}-\boldsymbol{y}'$ is orthogonal to two linearly independent tangent vectors on the unit sphere in $\boldsymbol{p}$ at the stationary point. Therefore, $\hat{\boldsymbol{p}}$ and $\boldsymbol{y}-\boldsymbol{y}'$ are colinear or anti-colinear. Consequently, $\hat{\boldsymbol{p}}$, $\hat{\boldsymbol{n}}$, and $\boldsymbol{y}-\boldsymbol{y}'$ must all line up.

Given $\boldsymbol{y}$, drop a perpendicular to $S_{y'}$. This determines a point, $\boldsymbol{y}'$, on $S_{y'}$ and a corresponding pair, σ_1 and σ_2. Furthermore, for this point, $\hat{\boldsymbol{n}}$ and $\boldsymbol{y}-\boldsymbol{y}'$ line up. Now find $\hat{\boldsymbol{p}}$ to line up with these two vectors. This determines a choice of θ and ϕ. In this manner, σ_1, σ_2, θ, and ϕ are all determined as functions of $\boldsymbol{y}$. Given the stationary four-tuple $(\sigma_1,\sigma_2,\theta,\phi)$, a second set, $(\sigma_1,\sigma_2,\pi-\theta,\pi-\phi)$, also satisfies the stationarity conditions, merely replacing the vector $\hat{\boldsymbol{p}}$ in the stationary set of vectors by $-\hat{\boldsymbol{p}}$. When $\hat{\boldsymbol{p}}$ is a direction in D_p, $-\hat{\boldsymbol{p}}$ is, as well. Thus, there may be none, one, or more than one such pairs of stationary points for a given choice of $\boldsymbol{y}$. Those values of $\boldsymbol{y}$ for which there are no stationary points are points at which $I_0(\lambda,\boldsymbol{y})$ (and, therefore, $I(\lambda,\boldsymbol{y})$, as well) is asymptotically of lower order than for those points where there are stationary pairs of four-tuples. Stationary points exist when the aperture in $\hat{\boldsymbol{p}}$ contains the normal from $\boldsymbol{y}$ to one or more points on $S_{y'}$.

The method of stationary phase requires the computation of the determinant and signature of a certain matrix, the Hessian of the phase function. For simplicity of notation, introduce

$$\theta = \sigma_3\,,\quad \phi = \sigma_4\,, \tag{15}$$

and then define the matrix

$$\left[\Phi_{ij}\right] = \left[\frac{\partial^2\Phi}{\partial\sigma_i\,\partial\sigma_j}\right],\quad i\,,\,j = 1-4\quad . \tag{16}$$

Here, all derivatives are to be evaluated at the stationary point. This calculation is carried out in Appendix A. The result is

$$I(\lambda, \boldsymbol{y}) \sim \sum G(\boldsymbol{y}, \boldsymbol{y}', \hat{\boldsymbol{p}})\; J(\lambda, \boldsymbol{y}) \;\; . \tag{17}$$

In this equation, the summation is over stationary points and $\boldsymbol{y}'$, $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{p}}$ are to be evaluated at each stationary point. The function $J(\lambda, \boldsymbol{y})$ is given by

$$J(\lambda, \boldsymbol{y}) = \frac{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{p}}}{2\pi} \int \frac{dp}{ip} \exp\left[i\lambda p \mu_3 \; |\boldsymbol{y} - \boldsymbol{y}'| + i\mu\pi/4 \right] \; , \tag{18}$$

with μ and μ_3 defined in equation (20), below. The domain of integration is the portions of the rays in the stationary directions, $\pm\hat{\boldsymbol{p}}$, which lie in D_p. The function G is given by

$$G(\boldsymbol{y}, \boldsymbol{y}', \hat{\boldsymbol{p}}) = \frac{f(\boldsymbol{y}, \hat{\boldsymbol{p}})}{\left[\left| \; 1 - \mu_1 \mu_3 \kappa_1 \; |\boldsymbol{y} - \boldsymbol{y}'| \; \right| \; \left| \; 1 - \mu_2 \mu_3 \kappa_2 \; |\boldsymbol{y} - \boldsymbol{y}'| \; \right| \right]^{1/2}} \; \cdot \tag{19}$$

Here, $\boldsymbol{\kappa}_j, \; j = 1{,}2$, are the principal curvature vectors on $S_{y'}$ at the stationary point and $\kappa_j, \; j = 1{,}2$, are their magnitudes. Furthermore,

$$\mu_1 = sgn\, \hat{\boldsymbol{p}} \cdot \boldsymbol{\kappa}_1 \, , \; \mu_2 = sgn\, \hat{\boldsymbol{p}} \cdot \boldsymbol{\kappa}_2 \, , \; \mu_3 = sgn\, \hat{\boldsymbol{p}} \cdot (\boldsymbol{y} - \boldsymbol{y}') \; , \; \mu = sgn \left[\Phi_{ij} \right] \; . \tag{20}$$

In carrying out the sum over stationary points, note that when $\boldsymbol{p}$ is replaced by $-\boldsymbol{p}$, μ_3 changes sign, so that the exponent changes sign. Furthermore, the factor $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{p}}/p$ changes sign, so that the quotient, $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{p}}/p$ changes sign. Thus, these two integrals add up to yield an integral with a signed variable p on both positive and negative parts of the real line. Below, let us assume that this pairing has occurred. The integration is to be carried out over the two intervals simultaneously and the summation is over only half of the stationary points, say the ones for which $\hat{p}_3 > 0$. When $p_3 = 0$, some equivalent ordering in $\hat{p}_1$ or $\hat{p}_2$ must be used.

$\boldsymbol{y}$ on $S_{y'}$

Consider the integral in (18) when $\boldsymbol{y}$ approaches the surface $S_{y'}$. Then, for the nearest stationary point, the phase approaches zero in this limit. (There may be other stationary points further away on $S_{y'}$ for which the limit is not zero.) For this distinguished stationary point, $J(\lambda, \boldsymbol{y})$ is $O(1)$ in λ in this limit, whereas it is $O(1/\lambda)$ for all other stationary points. Thus, $J(\lambda, \boldsymbol{y})$ changes its order in λ when $\boldsymbol{y}$ moves to $S_{y'}$. Therefore, it is reasonable to expect the sum to be dominated by this nearest stationary point.

It is shown in Appendix A that for $\boldsymbol{y}$ near $S_{y'}$, μ at this distinguished stationary point is equal to zero. Then J can be recognized as a band-limited step function. The magnitude of the step can be determined by evaluating $G(\boldsymbol{y},\boldsymbol{y}',\hat{\boldsymbol{p}})$ in the limit when $\boldsymbol{y}$ is on $S_{y'}$. The result is

$$G_{\text{peak}} = f(\boldsymbol{y},\hat{\boldsymbol{p}}),\ \boldsymbol{y} \text{ on } S_{y'} \ . \tag{21}$$

That is, the magnitude of the step is just the value of $f(\boldsymbol{y},\hat{\boldsymbol{p}})$, which is the magnitude of the discontinuity of f across the surface $S_{y'}$ at the stationary value of $\hat{\boldsymbol{p}}$. For two integrals over domains sharing the same segment of boundary surface, the stationary point is shared as well. In fact, the only differences in evaluation of the integral arise from the different values of the amplitude function, $G(\boldsymbol{y},\boldsymbol{y}',\hat{\boldsymbol{p}})$ in the two integrands and the fact that $\hat{\boldsymbol{n}}$ has opposite direction in the two integrals. Thus, the sum of the integrals will yield a difference of function values which reduces to just the jump in the function f when $\boldsymbol{y}$ is actually on the surface $S_{y'}$ and for the same value of $\hat{\boldsymbol{p}}$. This is the main result regarding the asymptotic identity operator, (3).

It should be noted that if the domain D_p does not contain the stationary value $\hat{\boldsymbol{p}} = \hat{\boldsymbol{n}}$, that is, if the normal to $S_{y'}$ at a particular $\boldsymbol{y}$ is not a direction in D_p, then there is no stationary point. In this case, the asymptotic expansion of I is lower order and, presumably, smaller in magnitude. This will be demonstrated in the next section through numerical examples.

Example: Aperture Limited Fourier Inversion of a Step Function

As a simple example of this type, consider the function

$$f(\boldsymbol{y}',\hat{\boldsymbol{p}}) = \begin{bmatrix} A, & 1 \leq y_3' \leq 2, \\ & \\ 0, & \text{otherwise.} \end{bmatrix} \tag{22}$$

Assume that $D_{y'}$ is the support of f and that D_p is a domain symmetric with respect to the origin that intersects the line $p_1 = p_2 = 0$ with $p \geq 1$ on D_p. Call that intersection D_3. By substituting this function into the definition (3) of $I(\lambda,\boldsymbol{y})$, it is fairly straightforward to obtain the result

$$I(\lambda,\boldsymbol{y}) = \frac{A}{2\pi}\int_{D_3} \frac{dp_3}{ip_3}\left[\exp\left[i\lambda p_3\ (y_3-1)\right] - \exp\left[i\lambda p_3(y_3-2)\right]\right] \ . \tag{23}$$

The integral here can be recognized as a difference of band-limited step functions which define the support of the function $f(\boldsymbol{y}',\hat{\boldsymbol{p}})$ in the y'-domain.

For this example, μ, defined in (20), is identically zero, as are the principal curvatures at both boundaries. Furthermore, $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{p}} = \pm 1$ at the two boundaries, accounting for the difference in sign in the two terms of the integrand. Finally, note that the effect of integration in $\boldsymbol{y}'$, followed by p_1 and p_2 is to evaluate $\hat{\boldsymbol{p}} = (0,0,\pm 1)$, that is, colinear or anti-colinear with the normals to boundary of the support of $f(\boldsymbol{y}', \hat{\boldsymbol{p}})$. Thus, if we were to introduce a multiplier, $g(\hat{\boldsymbol{p}})$ into the definition of f, all of the integrations could still be carried out to yield (23), except that now the integrand would have an additional factor of $g(0,0,1)$ as predicted by the theory. In summary, the asymptotic result is exact for this simple example.

Extracting Information about f on $S_{y'}$

In applications of this analysis to inverse problems, the aperture limited information about $f(\boldsymbol{x}'/L, \hat{\boldsymbol{k}})$ *in the Fourier domain* is the known data and the objective is to extract information about the function $f(\boldsymbol{x}/L, \hat{\boldsymbol{k}})$ in the spatial domain. The analysis presented here suggests that from large wave number aperture-limited data, one can, at best, expect to extract information about the boundary values of $f(\boldsymbol{x}/L, \hat{\boldsymbol{k}})$ at a value of $\hat{\boldsymbol{k}}$ distinguished by the fact that it is colinear with the normal to the boundary passing through $\boldsymbol{x}$. More generally, one might hope to determine the value of the jump in $f(\boldsymbol{x}/L, \hat{\boldsymbol{k}})$ across its discontinuity surfaces in the x-domain at the distinguished $\hat{\boldsymbol{k}}$ value.

There are two issues to be addressed: first, how one might most easily extract that information about f. In practice, band limited step functions are not easy to recognize. That is, given numerical output rather than an analytic expression for the band-limited step, the location of the step and its magnitude are not easily extracted, nor is information about the normal to the discontinuity surface of f. In fact, the amplitude of the band-limited step function is actually equal to zero right at its discontinuity, significantly mitigating the effect of growth of the integral I from $O(1/\lambda)$ to $O(1)$ in this region. Thus, searching for the midpoint of a band-limited step would not seem to be the most desirable approach. The second issue to be addressed is how one could go about determining the distinguished value of $\hat{\boldsymbol{p}} = \hat{\boldsymbol{k}}$ without having to actually construct the discontinuity surface. These issues will be addressed in the order presented. However, note that neither of the solutions to these problems is new. The former problem was first discussed in Bleistein and Cohen, [1979] and the latter was addressed in Bleistein, [1987a,b].

Processing for a Scaled Singular Function of the Boundary Surface, $S_{y'}$

The singular function of a surface, to be denoted by $\gamma(\boldsymbol{x})$, is a Dirac delta function with support on the given surface. Thus, it is a distribution having the property

$$\int_{-\infty}^{\infty} \gamma(\boldsymbol{x})\ G(\boldsymbol{x})\ d^3x = \int_{S_\gamma} G(\boldsymbol{x})\ dS \ , \tag{24}$$

Here, S_γ is the surface having $\gamma(\boldsymbol{x})$ as its singular function. Knowledge of $\gamma(\boldsymbol{x})$ constitutes mathematical imaging of the surface S_γ. A graphical display of $\gamma(\boldsymbol{x})$ provides an actual image of the surface.

With minor modification of the kernel, the integral operator in (1) can be transformed into one which the singular function of the boundary surface as its output, within a scale factor which will provide information about the boundary values of the function $f(\boldsymbol{y},\hat{\boldsymbol{p}})$.

What we need to transform the integral $J(\lambda,\boldsymbol{y})$ in (18) into a band-limited delta function is a multiplier which cancels the factor $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}/ip$ of the integrand. Thus, we need a multiplier of ip on half of the p-domain and $-ip$ on the image of that domain through the origin. The multiplier, $ik\, sgn\, \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{k}}$ with $\hat{\boldsymbol{u}}$ a constant[1] unit vector, will do the trick. When this factor is inserted into the integrand in (1), it will have exactly the desired effect. After rescaling,

$$ik\, sgn\, \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{k}} = i\lambda p\, sgn\, \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}} \ . \tag{25}$$

When D_k contains all directions, $\hat{\boldsymbol{k}}$, this multiplier is not defined in the plane of directions through the origin with $\hat{\boldsymbol{u}}$ as normal. In most applications, however, there is at least one plane of directions in which there is no information. That is, the domain D_k will exclude some plane through the origin. Choose $\hat{\boldsymbol{u}}$ as the unit normal to that plane.

Proceeding with this idea, in place of the integral (1), consider the following:

$$\bar{I} = \frac{i}{(2\pi)^3}\int_{D_k} k\, sgn\, \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{k}}\, d^3k \int_{D_{x'}} d^3x'\, f(\boldsymbol{x}'/L)\ \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')\right] \ . \tag{26}$$

[1]There is no reason why $\hat{\boldsymbol{u}}$ must be a constant vector. I simply have not found a need to make $\hat{\boldsymbol{u}}$ variable.

After rescaling according to (2) and (4), one obtains in place of (3) the integral,

$$\bar{I}(\lambda, \boldsymbol{y}) = \frac{i\lambda^4}{(2\pi)^3} \int_{D_p} p^2 \, sgn\, \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{p}} \, dp \sin\theta \, d\theta \, d\phi$$

$$\cdot \int_{D_{y'}} d^3y' \, f(\boldsymbol{y}', \hat{\boldsymbol{p}}) \exp\left[i\lambda p \, \hat{\boldsymbol{p}} \cdot (\boldsymbol{y} - \boldsymbol{y}')\right] \ , \tag{27}$$

and the integral I_0 in (8) and (9) is replaced by

$$\bar{I}_0(\lambda, \boldsymbol{y}) = \left[\frac{\lambda}{2\pi}\right]^3 \int_{D_p} p^2 \, sgn\, \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{p}} \, dp \sin\theta \, d\theta \, d\phi$$

$$\cdot \int_{S_{y'}} dS_{y'} \, \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{p}} \, f(\boldsymbol{y}', \hat{\boldsymbol{p}}) \exp\left[i\lambda p \Phi\right] \ . \tag{28}$$

The asymptotic analysis of this integral proceeds as for I_0, yielding in place of (17) and (18),

$$\bar{I}(\lambda, \boldsymbol{y}) \sim \sum G(\boldsymbol{y}, \boldsymbol{y}', \hat{\boldsymbol{p}}) \, \bar{J}(\lambda, \boldsymbol{y}) \ , \tag{29}$$

$$\bar{J}(\lambda, \boldsymbol{y}) = \frac{\lambda\mu_4}{2\pi} \int dp \exp\left[i\mu_3\lambda p \ |\boldsymbol{y} - \boldsymbol{y}'| + i\mu\pi/4\right], \ \ \mu_4 = \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{p}} \, sgn\, \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{p}} \ . \tag{30}$$

Except for the new parameter, μ_4, (± 1), the constituent functions and parameters here are still defined by (19) and (20).

For $\boldsymbol{y}$ on $S_{y'}$, there is a stationary point at $\boldsymbol{y}' = \boldsymbol{y}$ as long as $\hat{\boldsymbol{n}}$ is a direction in D_k. For this stationary point, $\bar{J} = O(\lambda)$, whereas for all other stationary points, or for $\boldsymbol{y}$ not on $S_{y'}$, $\bar{J} = O(1)$ in λ. Thus, as $\boldsymbol{y}$ approaches $S_{y'}$, one stationary point dominates the value of $I(\lambda, \boldsymbol{y})$ and the function value is larger by $O(\lambda)$ than the value for $\boldsymbol{y}$ bounded away from $S_{y'}$. As noted earlier, for this distinguished stationary point, $\mu = 0$, at least for $\boldsymbol{y}$ near $S_{y'}$. Recalling that we must sum over the two stationary points, $\pm\hat{\boldsymbol{p}}$, the integral J becomes

$$\bar{J}(\lambda, \boldsymbol{y}) = \frac{\lambda\mu_4}{2\pi} \int_{-p_+}^{-p_-} + \int_{p_-}^{p_+} dp \exp\left[i\lambda p \ |\boldsymbol{y} - \boldsymbol{y}'|\right] \ . \tag{31}$$

The limits of integration, p_- and p_+ are the intersections of the ray from the origin in the stationary direction $\hat{p}$ with the domain, D_p. In the k-domain, that is, undoing the scaling defined by (23),

$$\bar{J}(\lambda, \boldsymbol{y}) = \frac{2\mu_4}{2\pi} \int_{-k_+}^{-k_-} + \int_{k_-}^{k_+} dk \exp\left[ik \mid \boldsymbol{x} - \boldsymbol{x}' \mid\right] \ . \tag{32}$$

Here, $k_\pm$ have definitions completely analogous to $p_\pm$. In either form, $\bar{J}$ can be recognized as a symmetric (zero phase) band limited delta function with support at $|\boldsymbol{y} - \boldsymbol{y}'| = 0$ [equivalently, $|\boldsymbol{x} - \boldsymbol{x}'| = 0$], which occurs when $\boldsymbol{y}$ is on $S_{y'}$ [or $\boldsymbol{x}$ is on an equivalent surface in the x-domain, $S_{x'}$]. In this limit, the dominant term in (29) is readily evaluated with the aid of (21):

$$\bar{I}(\lambda, \boldsymbol{y}) \sim \lambda(p_+ - p_-)\,\mu_4\, f(\boldsymbol{y}, \hat{\boldsymbol{p}})/\pi = (k_+ - k_-)\,\mu_4\, f(\boldsymbol{x}/L, \hat{\boldsymbol{k}})/\pi\,,\ \boldsymbol{y} \text{ on } S_y\ . \tag{33}$$

Thus, the value of $\bar{I}$ on $S_{y'}$ is proportional to the interval width in the k-domain along an appropriate ray, multiplied by $\mu_4 f(\boldsymbol{x}/L, \hat{\boldsymbol{k}})/\pi$. If the integrand contained a filter factor, $F(\boldsymbol{k}) = F(K\boldsymbol{p})$, then the factor $p_+ - p_-$ is replaced by the area under the filter function in the stationary direction of $\hat{\boldsymbol{p}}$.

Qualitatively, for $\boldsymbol{x}$ near $S_{x'}$, the dominant term in the sum in (29) has the form of an aperture-limited singular function scaled by a slowly varying function. The scale factor becomes the jump in the function f multiplied by μ_4 when $\boldsymbol{x}$ is on $S_{x'}$ and the aperture-limited singular function becomes the area under the band pass filter in a distinguished direction, divided by 2π.

Within a sign, this output is the leading order high frequency band limited asymptotic expansion of the normal derivative of the discontinuous function, $f(\boldsymbol{y}, \hat{\boldsymbol{p}})$ in the x-domain. We obtain this result ***even though we use no*** a priori ***knowledge about the boundary surface or its normal direction in the processing operator, $\bar{I}$.***

Example: Processing for Singular Function Output

We return to the example (22). However, now, instead of applying the operator (1), we apply (26). At the same level of computation as in (23), the output is now

$$\bar{I}(\lambda, \boldsymbol{y}) = \frac{A\lambda\, sgn\, u_3}{2\pi} \int_{D_3} dp_3 \left[\exp\left[i\lambda p_3(y_3 - 1)\right] - \exp\left[i\lambda p_3(y_3 - 2)\right]\right]$$

$$= \frac{A\ sgn\ u_3}{\pi} \left[\frac{\sin\left[\lambda p_3(y-1)\right]}{y_3 - 1} - \frac{\sin\left[\lambda p_3(y_3-2)\right]}{y_3 - 2} \right] \Bigg|_{p_3=p_{3-}}^{p_{3+}} \quad . \tag{34}$$

One can see here that $\bar{I}(\lambda, \mathbf{y})$ is a difference of band limited delta functions that peak on the boundary surfaces of $D_{y'}$. That is, the output is proportional to the band limited singular functions of the boundary surface(s), $S_{y'}$. The scaling factor is the jump in f, namely, A multiplied by $\pm\mu_3/\pi$. The choice of sign here comes from the fact that the outward normal to the boundary has opposite direction on the two portions of the boundary surface.

For λp_{3-} "large" say, at least 3, the peaks of the two singlular functions will be well separated. If, in addition, the percentage bandwidth, $(p_{3+} - p_{3-})/(p_{3+} + p_{3-})$, is "large," say bigger than .5, the side lobes of the sinc functions appearing in (34) will be well damped and the boundaries will be well delineated by the band limited singular functions. The peak values of $\bar{I}$ are given by

$$\bar{I}(\lambda, x_1, x_2, 1) = \frac{A\lambda\ sgn\ u_3}{\pi}(p_{3+} - p_{3-}) = \frac{A\ sgn\ u_3}{\pi}(k_{3+} - k_{3-})$$

$$\bar{I}(\lambda, x_1, x_2, 2) = -\frac{A\lambda\ sgn\ u_3}{\pi}(p_{3+} - p_{3-}) = -\frac{A\ sgn\ u_3}{\pi}(k_{3+} - k_{3-}) \quad . \tag{35}$$

Thus, the modified operator, (26), produces a result from which the boundary surfaces and the amplitude of the discontiuity of f in the x-domain is more readily determined than from the ordinary Fourier inversion in (1).

It should be noted again that if the aperture D_p [equivalently, D_k] did not contain some segment of the line $p_1 = p_2 = 0$, then $I(\lambda, \mathbf{y}) = 0$. Since this line is the direction of the normal to the discontinuity surface of f, this result is consistent with the theory.

Example: Aperture Limited Fourier Inversion for a Ramp

The next example will show two features of aperture-limited large wave number Fourier inversion that are predicted by our results. Consider the function

$$f(\mathbf{y}', \hat{p}) = \begin{cases} A(1 - y'_3), & 0 \le y'_3 \le 1 \\ 0 \quad , & \text{otherwise} \ . \end{cases} \tag{36}$$

with $D_{y'}$ the support of $f(\mathbf{y}', \hat{\mathbf{p}})$.

It is straightforward to carry out all of the integrations of the operator I defined by (1) except p_3 to obtain the result

$$I(\lambda, \mathbf{y}) = \frac{A}{2\pi} \int_{D_3} \frac{dp_3}{ip_3} \left\{ \exp\left[i\lambda p_3\, y_3\right] \right.$$

$$\left. + \frac{1}{i\lambda p_3} \left[\exp\left[i\lambda p_3 (y_3 - 1)\right] - \exp\left[i\lambda p_3\, y_3\right] \right] \right\} . \tag{37}$$

If D_3 were the entire line, then the Fourier transform of the first exponential would be a step with support, $y_3 > 0$. The next two terms, which correct this step to produce the appropriate finite ramp, are lower order, $O(1/\lambda)$, for large wave number aperture-limited data. Note that a phase depending on the difference, $y_3 - 1$, arises only in this lower order term. The reason for this is that the function defined by (36) is continuous at this boundary.

From aperture-limited large wave number data, one should not expect to detect the lower order, $O(1/\lambda)$, contribution to the Fourier transform of $f(\mathbf{y}, \hat{\mathbf{p}})$, but only the leading order term. To leading order, then, $I(\lambda, \mathbf{y})$ is a band limited step function with discontinuity on the discontinuity surface of $f(\mathbf{y}, \hat{\mathbf{p}})$ and amplitude equal to the amplitude of the discontinuity of $f(\mathbf{y}, \hat{\mathbf{p}})$. We could use the operator, $\bar{I}(\lambda, \mathbf{y})$, defined by (26) or (27), to more easily detect the location of this discontinuity surface and the value, A. In this case, the leading order step would be replaced by a leading order singular function of the boundary surface, $y_3 = 0$, while the second order ramp would be replaced by a second order step on the interval, $0 \le y_3 \le 1$.

Of course, one might consider using yet another operator, with multiplier $-\lambda^2 p^2$ to detect the jump in the first derivative of $f(\mathbf{y}, \hat{\mathbf{p}})$. While this will work for synthetic examples, it would be considerably less reliable with field data and "many" discontinuity surfaces of $f(\mathbf{y}, \hat{\mathbf{p}})$ and its derivatives.

Determination of $\hat{\mathbf{p}}$ at the Distinguished Stationary Point

We will show here how to determine the distinguished value of $\hat{\mathbf{p}}$ at the stationary point by direct computation of an operator output. We consider this method superior to determination of the distinguished $\hat{\mathbf{p}}$ graphically from the image of the boundary surface. This method has been used successfully by Parsons [1986] and Sullivan and Cohen [1987].

The determination of $\hat{\mathbf{p}}$ is equivalent to the determination of $\sin\theta$ or $\cos\theta$ and $\sin\phi$ or $\cos\phi$ at the stationary point. To find these angles, note first from (4) that

$$\cos\theta = \frac{k_3}{k} = \frac{p_3}{p} \ , \ \cos\phi = \frac{k_1}{\sqrt{k_1^2 + k_2^2}} = \frac{p_1}{\sqrt{p_1^2 + p_2^2}} \quad . \tag{38}$$

Suppose that we define two new integral operators with these factors introduced into the kernels of the operator, $\overline{I}$. That is, starting from (26), define

$$\overline{I}_\theta = \int_{D_k} k_3 \, sgn\, \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{k}} \, d^3k \int_{D_{x'}} d^3x' \, f(\boldsymbol{x}'/L, \hat{\boldsymbol{k}}) \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')\right] \ , \tag{39}$$

$$\overline{I}_\phi = \int_{D_k} \frac{k\, k_1 \, sgn\, k_3}{\sqrt{k_1^2 + k_2^2}} \, d^3k \int_{D_{x'}} d^3x' \, f\,(\boldsymbol{x}'/L, \hat{\boldsymbol{k}}) \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')\right] \ . \tag{40}$$

Let us now consider the asymptotic analysis of $\overline{I}_\theta$ and $\overline{I}_\phi$. Clearly, it will proceed exactly as above, with only the amplitudes of the results being affected by the changes introduced here. Thus, the peak values of these new integral operators can also be predicted. They will differ from the result for $\overline{I}$, itself, by the factors, $\cos\theta$, $\cos\phi$, ***each evaluated at the distinguished stationary point.*** That is,

$$\cos\theta = \frac{\overline{I}_{\theta\ \mathrm{peak}}}{I_{\ \mathrm{peak}}} \ , \ \cos\phi = \frac{\overline{I}_{\phi\ \mathrm{peak}}}{I_{\ \mathrm{peak}}} \ , \ \boldsymbol{y} \text{ on } S_{y'} \quad . \tag{41}$$

The function, $\sin\phi$, can be determined this way as well, while $\sin\theta$ can be defined in terms of $\cos\theta$, taking the positive square root $(0 \leq \theta \leq \pi)$. Given these values at the stationary point, $\hat{\boldsymbol{p}}$ is determined there.

Integrands with Other Types of Singularities

It should be noted that the multiplier, $ik\, sgn\, \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{k}}$, is a "best choice" only because the properties of the function, $f\,(\boldsymbol{x}'/L, \hat{\boldsymbol{k}})$, namely, that this function was assumed to be smooth but not to vanish smoothly on the boundary of its domain of definition. In particular, the original operator, I, will optimally depict a function which is a (sum of) Dirac delta function(s). Furthermore, $I(\boldsymbol{x})$ will also be a better operator than $\overline{I}$ for singular functions. That can readily be verified now, as follows. Let us consider the application of I as defined by (1) to the function,

$$f(\boldsymbol{x}'/L, \hat{\boldsymbol{k}}\,) = \gamma(\boldsymbol{x}'/L) \ , \tag{42}$$

with $\gamma(\boldsymbol{y}')$, the singular function of a surface, S, as defined in equation (24) and the related discussion. Substitution of this function into (1) and use of the change of variables defined by (2) and (4) leads to the result

$$I(\lambda, \boldsymbol{y}) = \left[\frac{\lambda}{2\pi}\right]^3 \int_{D_p} p^2 \, dp \sin\theta \, d\theta \, d\phi \int_S dS \exp\left[i\lambda p \Phi\right] \quad , \tag{43}$$

with Φ defined by (5).

Thus, without introducing an additional multiplier to modify the operator I to $\overline{I}$, this integral is exactly like the integral $\overline{I}_0(\lambda, \boldsymbol{y})$ defined by (28), with the amplitude of that integral, $sgn\,(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}})\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}\,f\,(\boldsymbol{y}',\hat{\boldsymbol{p}}) = \mu_4 f\,(\boldsymbol{y}',\hat{\boldsymbol{p}})$ replaced by unity. Consequently, the asymptotic analysis for this integral has already been done. The output will be an aperture limited version of $\gamma(\boldsymbol{y})$. That is, it will be a band limited delta function along each normal which is a direction from the origin to D_k. The structure of the delta function will depend on the extent of that ray in the k-domain and also on any smoothing that might be applied through the introduction of a high wave number band pass filter along that ray.

Asymptotically, then, the high wave number aperture limited Fourier inversion of data for a singular function behaves just as the exact result did for a planar surface.

Minor Extension

In the applications of interest, the function, f, may depend on $\boldsymbol{x}$, in addition to the other dependencies indicated in (1). This does not change the results stated here, since $\boldsymbol{x}$ merely acts as a parameter with respect to the integrations in (1).

A Form That Arises in the Analysis of Kirchhoff Data for the Inverse Scattering Formalism

A last extension to be considered here is the case in which the integrand has both a singular function and a smoother amplitude function. Namely, in (1), replace $f\,(\boldsymbol{x}'/L, \hat{\boldsymbol{k}})$ by $f\,(\boldsymbol{x}'/L, \boldsymbol{x}/L, \hat{\boldsymbol{k}})\gamma(\boldsymbol{x}'/L)$. Then, after exploiting the singular function, the integral (1) becomes

$$I = \frac{1}{(2\pi)^3} \int_{D_k} d^3k \int_{S_{x'}} dS' \, f(\boldsymbol{x}'/L, \boldsymbol{x}/L, \hat{\boldsymbol{k}}) \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')\right] \quad , \tag{44}$$

or, in dimensionless variables,

$$I = \left[\frac{\lambda}{2\pi}\right]^3 \int_{D_p} d^3p \int_{S_{y'}} dS_{y'}\, f(\mathbf{y}', \mathbf{y}, \hat{p}) \exp\left[i\lambda p\Phi\right] \quad . \tag{45}$$

Here, Φ is defined by (5). Except for the fact that we have not used polar coordinates, the integral is of the same form as (28) with the amplitude $\hat{\mathbf{n}}\cdot\hat{p}\, sgn\, \hat{\mathbf{u}}\cdot\hat{\mathbf{p}} f(\mathbf{y}',\hat{p}) = \mu_4 f(\mathbf{y}',\hat{p})$ replaced by $f(\mathbf{y}',\mathbf{y},\hat{p})$. The analysis proceeds as before, with the extra dependence on $\mathbf{y}$ in f merely playing the role of a parameter as regards the asymptotic analysis of the integral.

Consequently, the asymptotic expansion of the integral in (45) is given by (29), with the following insertions. The function $G(\mathbf{y},\mathbf{y}',\hat{\mathbf{p}})$ is defined by (19), except that $f(\mathbf{y}',\hat{p})$ is replaced by $f(\mathbf{y}',\mathbf{y},\hat{p})$ and $\bar{J}$ is given by (30), (31) or (36) with μ_4 replaced by unity. In particular, for $\mathbf{x}$ near S, the asymptotic expansion is dominated by a scaled band limited singular function. When $\mathbf{x}$ is on S, the scale factor is just the amplitude, $f(\mathbf{x}'/L, \mathbf{x}/L, \hat{k})$, evaluated at the stationary point and determinable as described in the discussion above for determination of $\hat{\mathbf{p}}$.

Summary

It has been shown here that the high wave number aperture limited Fourier inversion of the Fourier transform of a piecewise smooth function is dominated by the function values on the discontinuity surface(s). The inversion is approximately a band limited step function of normal distance from each point on the discontinuity surface. The amplitude of the step function is proportional to the jump in the function across the surface at the point in question. By modifying the inversion operator, the output can be transformed into the singular function(s) of the discontinuity surface(s) of the original function, again scaled by the jump in the original function at each point of the discontinuity surface. This modification facilitates the numerical identification of the discontinuity surface(s) and the amplitude of the jump at each point on the surface(s).

This aperture limited inversion of data was shown to produce an aperture limited approximation of the singular function.

In the language of asymptotic expansions of integrals, given an output point, $\mathbf{y}$, the high wave number aperture limited Fourier inversion of a piecewise smooth function or of the singular function of a surface is dominated by certain critical points which can be determined as functions of $\mathbf{y}$. A critical point consists of a location $\mathbf{y}'$ in the spatial domain and a direction $\hat{\mathbf{p}}$ in the dual Fourier domain. When $\mathbf{y}$ is on a discontinuity surface of the smooth function or on the support surface of the singular function and the normal to the surface in question is a direction $\hat{\mathbf{p}}$ in the aperture D_p, the output is at least $O(\lambda)$ larger than it is otherwise. This increase in order can be exploited to detect the discontinuity surface(s) of piecewise smooth functions or support surface(s) of singular functions and estimate an amplitude function on the support surface(s).

NUMERICAL EXAMPLES

We show here some numerical examples that demonstrate the theory. All are two dimensional, allowing us to use the third dimension to depict the amplitude of the output.

Aperture Limited Singular Function

As a first example, consider the singular function of the line, $x_1 = 0$. This function has as exact Fourier transform, $\delta(k_2)$. Thus, the aperture limited Fourier inversion will be nonzero only if D_k contains some segment of the line $k_2 = 0$. In the context of our theory, the normal to the support domain of the singular function is (0,1). Therefore, this must be a direction in D_k. When the converse is true, the theory predicts that the output will be asymptotically zero to leading order. In this simple example, the asymptotic result is exactly true.

Figure 1 depicts the singular function of the line, $x_1 = 0$. This depiction of the singular function extends over only one sample point and is therefore a full bandwidth singular function for the discrete (FFT) Fourier transform to be used below. The extra "spike" at the corner is shown for normalization purposes; it will appear in subsequent figures. Figure 2 shows the k-domain, D_k, to be used for inversion. Note that this domain includes the normal direction to the line, $x_1 = 0$ of Figure 1. Figure 3 shows the aperture limited inversion. The aperture limited singular function adequately identifies the original function. Figure 4 depicts another k-domain, D_k, for inversion of the same data. Note that this domain does *not* include the normal direction to the line, $x_1 = 0$, although, on a percentage basis, it is much larger than the previous domain. Figure 5 is a plot of the aperture-limited inversion for this domain, D_k, on the same scale as Figures 1 and 3. Figure 6 shows the same output, scaled up by six orders of magnitude.

The more general case is demonstrated in Figures 7-11. Figure 7 depicts the singular function of a circle. Figure 8 depicts a k-domain, D_k, in which the angle of $\boldsymbol{k}$ is not restricted, but only its magnitude. Figure 9 is the inversion, showing an aperture limited Dirac delta function along every radial line (every normal to the original curve). Figure 10 shows an alternative k-domain, D_k, for which the direction of $\boldsymbol{k}$ [equivalently, $\hat{\boldsymbol{k}}$] is restricted, as well. Figure 11 shows the inversion. Here, the singular function is adequately defined only for the normal to the circle in the angular aperture of D_k and the output falls off strongly to zero outside of that angular aperture.

Relevance to Inverse Scattering

This last example has relevance to inverse problems. The surface, S_γ, might be a reflector. Thus, determination of $\gamma(\boldsymbol{x}/L)$ from aperture-limited Fourier data constitutes mathematical imaging of the reflecting surface. In practice, the singular function of the surface might also be multiplied by a reflection coefficient which is a function of $\hat{\boldsymbol{k}}$. As noted above, $\hat{\boldsymbol{k}}$ coligns with the normal to S_γ and therefore it is fixed, as well. Because of the subtleties of the dependence of the reflection coefficient on $\hat{\boldsymbol{k}}$ in applications, the reflection coefficient that is determined need not be the normal reflection coefficient. See Bleistein [1987a, 1987b] and the discussion of the next section.

APERTURE LIMITED INVERSE SCATTERING

From references cited here, as well as many others, the evidence is overwhelming that high-frequency inverse scattering is closely related to aperture-limited Fourier inversion. However, it is not always true that high-frequency inversion is equivalent to large wave number inversion. See, for example, Beylkin, Oristaglio and Miller [1985]; Miller, Oristaglio and Beylkin [1987]; and Mora [1987]. For simple scattering experiments in which the sources and receivers are on the "same side" of the scattering region, high frequency does, indeed, correspond to large wave number. That is the type of experiment of interest here.

The inversion operator in equation (49), below, was proposed by the author [Bleistein, 1987b] for an inverse scattering problem with the following properties:

(1) a variable background sound speed and density for back propagation are prescribed;

(2) a source receiver array, such as (a) common (or single) source, multi-receiver, (b) common receiver, multi-source, (c) common offset between source and receiver, is prescribed;

(3) the data can be described as high frequency data.

Under further conditions to be discussed below, the operator was shown to yield a reflector map and a means of estimating parameter changes across the reflector. In this section, we show how analysis of the application of the inversion operator to model data is related to the discussion of large wave number aperture limited Fourier inversion as discussed in the previous two sections.

The Inverse Problem

Consider an inverse scattering problem for acoustic wave propagation in which the source/receiver pairs, $\boldsymbol{x}_s$ and $\boldsymbol{x}_r$, respectively, are identified by a parameter $\boldsymbol{\xi}=(\xi_1,\xi_2)$ as follows:

$$\boldsymbol{x}_s = \boldsymbol{x}_s(\boldsymbol{\xi}),\ \boldsymbol{x}_r = \boldsymbol{x}_r(\boldsymbol{\xi}) \quad . \tag{46}$$

For example, in the case of a common (or single) source experiment, $\boldsymbol{x}_s$ would be a constant vector denoting that fixed position and the function $\boldsymbol{x}_r(\boldsymbol{\xi})$ would be a parametric representation of the receiver surface. For the common receiver case, the roles of $\boldsymbol{x}_s$ and $\boldsymbol{x}_r$ would be reversed. For the experiment with common (or fixed) offset (between source and receiver), both $\boldsymbol{x}_s$ and $\boldsymbol{x}_r$ would vary with $\boldsymbol{\xi}$.

It is assumed that the sound speed $c(\boldsymbol{x})$ and the density $\rho(\boldsymbol{x})$ are known down to a reflecting surface S; below S, these variables take on new values, $c_+(\boldsymbol{x})$ and $\rho_+(\boldsymbol{x})$, to be determined, along with the surface S, itself.

It is further assumed that $u(\boldsymbol{x},\boldsymbol{x}_s,\omega)$ is the response to an impulsive point source at $\boldsymbol{x}_s$. Above S, u satisfies the wave equation

$$\rho\nabla\cdot\left[\frac{1}{\rho}\,\nabla u\right]+\frac{\omega^2}{c^2}\,u=-\,\delta(\boldsymbol{x}-\boldsymbol{x}_s) \quad . \tag{47}$$

Below S, u satisfies this equation with right side zero and with ρ and c replaced by ρ_+ and c_+, respectively. Furthermore, u must satisfy two continuity conditions on S, namely that u and $1/\rho\,\partial u/\partial n$ are continuous across S.

The total solution above S may be viewed as the sum of a free-space Green's function and the upward scattered response to the reflector. The Green's function satisfies (47) everywhere, with $c(\boldsymbol{x})$ analytically continued throughout space as an infinitely smooth function.[2] Denote the second term by $u_S(\boldsymbol{x},\boldsymbol{x}_s,\omega)$. Thus, the data for the inverse scattering experiment is

$$D(\boldsymbol{\xi},\omega)=u_S\left(\boldsymbol{x}_r(\boldsymbol{\xi}),\,\boldsymbol{x}_s(\boldsymbol{\xi}),\,\omega\right) \quad . \tag{48}$$

In Bleistein [1987b], the following inversion operator to process this data was

[2]Clearly, there are extensions in which the reference is only "piecewise smooth" and the Green's function is replaced by a "primaries only" Green's function, or one that contains multiple reflections through interfaces above S.

introduced:

$$\beta(\boldsymbol{x}) = \frac{1}{8\pi^3} \int_{S_\xi}\int \left[\frac{\rho(\boldsymbol{x}_s)}{\rho(\boldsymbol{x}_r)}\right]^{1/2} d^2\xi \, \frac{|\, h(\boldsymbol{x},\xi)\,|\; N_1(\boldsymbol{x},\xi)}{A(\boldsymbol{x},\boldsymbol{x}_s)\, A(\boldsymbol{x},\boldsymbol{x}_r)\, \left|\nabla\tau(\boldsymbol{x},\boldsymbol{x}_s) + \nabla\tau(\boldsymbol{x},\boldsymbol{x}_r)\right|}$$

$$\cdot \int i\omega\, d\omega\, F(\omega) \exp\left[-i\omega\left[\tau(\boldsymbol{x},\boldsymbol{x}_s) + \tau(\boldsymbol{x},\boldsymbol{x}_r)\right]\right] D(\xi,\omega) \quad . \tag{49}$$

The domain of integration S_ξ is the set of ξ-values which are required to cover the source/receiver array. The domain of integration in ω is limited by the high pass filter $F(\omega)$. The functions $\tau(\boldsymbol{x},\boldsymbol{x}_s)$ and $A(\boldsymbol{x},\boldsymbol{x}_s)$ $[\tau(\boldsymbol{x},\boldsymbol{x}_r)$ and $A(\boldsymbol{x},\boldsymbol{x}_r)]$ are the WKBJ or ray-theoretic phase and amplitude of the constant density Green's function with source at $\boldsymbol{x}_s$ $[\boldsymbol{x}_r]$ and observation point $\boldsymbol{x}$.

The function, $h(\boldsymbol{x},\xi)$, is the essential element of this inversion formula. It is the determinant,

$$h(\boldsymbol{x},\xi) = \det \begin{vmatrix} \nabla\left[\tau(\boldsymbol{x},\boldsymbol{x}_s) + \tau(\boldsymbol{x},\boldsymbol{x}_r)\right] \\ \dfrac{\partial}{\partial\xi_1}\nabla\left[\tau(\boldsymbol{x},\boldsymbol{x}_s) + \tau(\boldsymbol{x},\boldsymbol{x}_r)\right] \\ \dfrac{\partial}{\partial\xi_2}\nabla\left[\tau(\boldsymbol{x},\boldsymbol{x}_s) + \tau(\boldsymbol{x},\boldsymbol{x}_r)\right] \end{vmatrix} \quad . \tag{50}$$

The factor, $N_1(\boldsymbol{x},\xi)$, was omitted in the above cited references. This is a neutralizer or cutoff function having the property that h is of one sign on the support of this function. The neutralizer is equal to unity except near the boundary of its support domain, where it decreases C^∞ smoothly to zero. The introduction of this function precludes spurious artifacts in the output arising from the boundary of S_ξ. Further discussion of the assumption, $h=0$, is provided in Beylkin [1985] and in Beylkin, Oristaglio and Miller [1985].

The derivation of this inversion formula starts from Beylkin's [1985] equation (4.3), expressing that result in the frequency domain. Asymptotic analysis indicates that for high frequency band limited data, Beylkin's inversion formula will produce a band-limited step function wherever the impedance has a discontinuity. That is, the output of the inversion formula is similar to the output of aperture limited Fourier inversion applied to piecewise smooth functions. Proceeding with an analysis equivalent to the determination of the modified Fourier inversion operator, $\bar{I}$, of the previous sections, leads to a modification of the Beylkin inversion operator that yields a singular function output instead of step function output. The key to this modification is the identification of an appropriate wave vector, $\boldsymbol{k}$, which is also indicated in Beylkin's [1985] paper.

In Bleistein [1987a,b], the validity of the modified inversion was confirmed by applying the operator to model data derived using the Kirchhoff approximation. The purpose of this section is to show that, in fact, the analysis can be reduced to the simpler problem of aperture limited Fourier inversion as discussed in the previous sections.

To show how this is done, the inversion formula is applied to the upward scattered response from a single reflector, S, modeled by the Kirchhoff approximation. This representation can be found in many sources, including Bleistein [1986], eq. (74). In the notation used here the result is

$$D(\xi,\omega) \sim i\omega\left[\frac{\rho(\boldsymbol{x}_r)}{\rho(\boldsymbol{x}_s)}\right]^{1/2}\int_S R(\boldsymbol{x}',\boldsymbol{x}_s)\,A(\boldsymbol{x}',\boldsymbol{x}_s)\,A(\boldsymbol{x}',\boldsymbol{x}_r)\,\hat{\boldsymbol{n}}\cdot\nabla'\left[\tau(\boldsymbol{x}',\boldsymbol{x}_s)+\tau(\boldsymbol{x}',\boldsymbol{x}_r)\right]$$

$$\cdot N_2(\boldsymbol{x}')\exp\left\{i\omega\left[\tau(\boldsymbol{x}_s,\boldsymbol{x})+\tau(\boldsymbol{x}_r,\boldsymbol{x})\right]\right\}dS' \quad . \tag{51}$$

In this equation, ∇' denotes a gradient with respect to the $\boldsymbol{x}'$ variables and $R(\boldsymbol{x}',\boldsymbol{x}_s)$ is the geometrical optics reflection coefficient,

$$R(\boldsymbol{x}',\boldsymbol{x}_s) =$$

$$\frac{\dfrac{1}{\rho(\boldsymbol{x}')}\left|\dfrac{\partial\tau(\boldsymbol{x}',\boldsymbol{x}_s)}{\partial n}\right| - \dfrac{1}{\rho_+(\boldsymbol{x}')}\left[\dfrac{1}{c_+^2(\boldsymbol{x}')}-\dfrac{1}{c^2(\boldsymbol{x}')}+\left[\dfrac{\partial\tau(\boldsymbol{x}',\boldsymbol{x}_s)}{\partial n}\right]^2\right]^{1/2}}{\dfrac{1}{\rho(\boldsymbol{x}')}\left|\dfrac{\partial\tau(\boldsymbol{x}',\boldsymbol{x}_s)}{\partial n}\right| + \dfrac{1}{\rho_+(\boldsymbol{x}')}\left[\dfrac{1}{c_+^2(\boldsymbol{x}')}-\dfrac{1}{c^2(\boldsymbol{x}')}+\left[\dfrac{\partial\tau(\boldsymbol{x}',\boldsymbol{x}_s)}{\partial n}\right]^2\right]^{1/2}} \quad . \tag{52}$$

The unit normal $\hat{\boldsymbol{n}}$ points upward and $\partial/\partial n = \hat{\boldsymbol{n}}\cdot\nabla'$; $N_2(\boldsymbol{x}')$ is a neutralizer function, equal to unity except in a neighborhood of the boundary of S. If S is of infinite extent, let $N_2(\boldsymbol{x}')$ vanish "sufficiently far" from the origin. The reason for this, is that, again, edge effects from the boundary of the domain of integration are not of interest in the current analysis. The result (51) is substituted into (49) to obtain the following multifold integral representation of the output $\beta(\boldsymbol{x})$ when applied to this synthetic data:

$$\beta(\boldsymbol{x}) \sim \frac{1}{8\pi^3} \int_{S_\xi} d^2\xi \, \frac{\left|h(\boldsymbol{x}, \xi)\right| N_1(\boldsymbol{x}, \xi)}{A(\boldsymbol{x}, \boldsymbol{x}_s)\, A(\boldsymbol{x}, \boldsymbol{x}_r) \left|\nabla\tau(\boldsymbol{x}, \boldsymbol{x}_s) + \nabla\tau(\boldsymbol{x}, \boldsymbol{x}_r)\right|} \int \omega^2 \, d\omega \, F(\omega)$$

$$\cdot \int_S R(\boldsymbol{x}', \boldsymbol{x}_s)\, A(\boldsymbol{x}', \boldsymbol{x}_s)\, A(\boldsymbol{x}', \boldsymbol{x}_r) \exp\left\{ i\omega\, \Phi(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}_s, \boldsymbol{x}_r) \right\}$$

$$\cdot N_2(\boldsymbol{x}')\, \hat{\boldsymbol{n}} \cdot \nabla' \left[\tau(\boldsymbol{x}', \boldsymbol{x}_s) + \tau(\boldsymbol{x}', \boldsymbol{x}_r)\right] dS' \quad . \tag{53}$$

In this equation,

$$\Phi(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}_s, \boldsymbol{x}_r) = \tau(\boldsymbol{x}', \boldsymbol{x}_s) + \tau(\boldsymbol{x}', \boldsymbol{x}_r) - \left[\tau(\boldsymbol{x}', \boldsymbol{x}_s) + \tau(\boldsymbol{x}', \boldsymbol{x}_r)\right] \tag{54}$$

is the difference of travel times, source point $[\boldsymbol{x}_s]$ to input point $[\boldsymbol{x}']$ to receiver point $[\boldsymbol{x}_r]$ minus source point to output point $[\boldsymbol{x}]$to receiver point. The surface S is described parametrically in terms of two parameters, σ_1, σ_2, by

$$\boldsymbol{x}' = \boldsymbol{x}'(\boldsymbol{\sigma}), \, \boldsymbol{\sigma} = (\sigma_1, \sigma_2) \quad . \tag{55}$$

In terms of these parameters,

$$dS' = \sqrt{g}\, d\sigma_1 \, d\sigma_2 \quad , \tag{56}$$

with g the first fundamental form of differential geometry for S in terms of the parameters $\boldsymbol{\sigma}$.

In Bleistein [1987a], the method of stationary phase was applied to the integral (53) in the variables ξ and $\boldsymbol{\sigma}$. The conditions of stationarity have the following interpretation. First, the point $\boldsymbol{x}'$ on S is such that Snell's law is satisfied by the rays from $\boldsymbol{x}_r$ and $\boldsymbol{x}_s$ to $\boldsymbol{x}'$. A particular source/receiver pair is picked out by the second condition which ties the triple, $\boldsymbol{x}'$, $\boldsymbol{x}_r$ and $\boldsymbol{x}_s$ to $\boldsymbol{x}$. This condition states that the sum of traveltime gradients from $\boldsymbol{x}'$ to $\boldsymbol{x}_r$ and $\boldsymbol{x}_s$ and the same sum from $\boldsymbol{x}$ must have equal projections on the observation surface defined by (46). For $\boldsymbol{x}$ on the reflector, S, $\boldsymbol{x}' = \boldsymbol{x}$ is a stationary point, with $\boldsymbol{x}_r$ and $\boldsymbol{x}_s$ being the source receiver pair whose ray trajectories satisfy Snell's law for reflection at this point. The condition, $h \neq 0$, guarantees that there is only one such point for sufficiently smooth surfaces, S. It is assumed that the surface under consideration is in that class. Clearly, then, for $\boldsymbol{x}$ in a neighborhood of S, there is still only one stationary point, having $\boldsymbol{x}' = \boldsymbol{x}$ as its limiting value as $\boldsymbol{x}$ approaches S.

Let us suppose, now, that $\boldsymbol{x}$ is near S. Introduce a neutralizer function, $N(\boldsymbol{x}',\boldsymbol{x})$, which is identically equal to unity in some disk, say $|\boldsymbol{x}' - \boldsymbol{x}| \le r_1$, identically equal to zero outside some larger disk, $|\boldsymbol{x}' - \boldsymbol{x}| \ge r_2 > r_1$, and is infinitely differentiable everywhere. Then the following is true.

Lemma 2:

Consider the integrals obtained from (53) by introducing $N(\boldsymbol{x}',\boldsymbol{x})$ and $N^*(\boldsymbol{x}',\boldsymbol{x}) = 1 - N(\boldsymbol{x}',\boldsymbol{x})$ as multipliers in the integrand. Call the first integral $\beta_N(\boldsymbol{x})$ and the second integral $\beta_N^*(\boldsymbol{x})$. Then $\beta_N^*(\boldsymbol{x}) = o(\omega^{-m})$, where m is limited only by the smoothness of the integrand.

The proof is given in Appendix B.

Now consider the integral, $\beta_N(\boldsymbol{x})$. This integral can be reduced to an aperture-limited Fourier identity operator of the type discussed in the previous section. To do so, first observe [Beylkin, 1985] that for $\boldsymbol{x}'$ near $\boldsymbol{x}$:

$$\omega\Phi(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}_s, \boldsymbol{x}_r) = \omega\left[\nabla_{\boldsymbol{x}}\left[\tau(\boldsymbol{x}, \boldsymbol{x}_r) + \tau(\boldsymbol{x}, \boldsymbol{x}_r)\right]\cdot(\boldsymbol{x}' - \boldsymbol{x}) + 0\left(|\boldsymbol{x}' - \boldsymbol{x}|^2\right)\right] \quad . \tag{57}$$

That is, the phase is nearly of the form $\boldsymbol{k}\cdot(\boldsymbol{x}' - \boldsymbol{x})$, with

$$\boldsymbol{k} = \omega\nabla_{\boldsymbol{x}}\left[\tau(\boldsymbol{x}, \boldsymbol{x}_s) + \tau(\boldsymbol{x}, \boldsymbol{x}_r)\right] \quad . \tag{58}$$

In fact,

Lemma 3:

In some neighborhood of $\boldsymbol{x}$ there exists a change of variables from $(\omega,\boldsymbol{\xi})$ to $\boldsymbol{k} = (k_1, k_2, k_3)$ with nonvanishing Jacobian, having the following properties:

$$\boldsymbol{k} = \omega\left[\nabla_{\boldsymbol{x}}\left[\tau(\boldsymbol{x}, \boldsymbol{x}_s) + \tau(\boldsymbol{x}, \boldsymbol{x}_r)\right] + O\left(|\boldsymbol{x}' - \boldsymbol{x}|\right)\right] \quad ; \tag{59}$$

and

$$\frac{\partial(\boldsymbol{k})}{\partial(\omega, \boldsymbol{\xi})} = \omega^2 H(\boldsymbol{x},\boldsymbol{\xi}) = \omega^2\left[h(\boldsymbol{x}, \boldsymbol{\xi}) + O\left(|\boldsymbol{x}' - \boldsymbol{x}|\right)\right] \quad . \tag{60}$$

The proof is given in Appendix C.

It is interesting to note that the regularity of the transformation does not depend on ω, but only on the spatial variables. Furthermore,

Lemma 4:

$\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ is a function of $\xi \, sgn \, \omega$ with nonvanishing Jacobian. Conversely, for each choice of $sgn \, \omega$, ξ is a function of $\hat{\boldsymbol{k}}$.

That is, if we were to pick two parameters, such as the first two components of $\hat{\boldsymbol{k}}$ or the polar angles of $\hat{\boldsymbol{k}}$ with respect to some set of axes, then these variables are functions of ξ (and not ω) with nonvanishing Jacobian wherever $H(\boldsymbol{x}, \xi)$ is nonvanishing, except for the sign of ω. This follows from the proof of Lemma 3.

This result has an important implication with regard to the aperture of $\boldsymbol{k}$ values in the domain of integration after transformation to these variables. The directions of the k-vectors in the domain of integration is solely a function of the source/receiver configuration. The approximate form of the transformation in (59) suggests further that this angular aperture is almost completely determined by the sum of the gradients of the travel times at $\boldsymbol{x}$. Equivalently, this is the sum of the tangents to the rays from the source and receiver. In Beylkin, Oristaglio and Miller [1985] and Miller, Oristaglio and Beylkin [1987], extensive use is made of this fact. For our own purposes, here, the significance of this observation is what it implies about the integrand of (53) after transformation. Namely, the integral takes the form

$$\beta(\boldsymbol{x}) = \frac{1}{8\pi^3} \int_{D_k} d^3k \, F(\omega(\boldsymbol{k})) \int_{S'} dS' \, f(\boldsymbol{x}', \hat{\boldsymbol{k}}) \exp\left[i\boldsymbol{k} \cdot (\boldsymbol{x}' - \boldsymbol{x}) \right] \quad . \tag{61}$$

In this equation,

$$f(\boldsymbol{x}', \boldsymbol{x}, \hat{\boldsymbol{k}}) = -R(\boldsymbol{x}', \boldsymbol{x}_s) \frac{A(\boldsymbol{x}', \boldsymbol{x}_s) \, A(\boldsymbol{x}', \boldsymbol{x}_r) \, \hat{\boldsymbol{n}} \cdot \nabla' \left[\tau(\boldsymbol{x}', \boldsymbol{x}_s) + \tau(\boldsymbol{x}', \boldsymbol{x}_r) \right] |h(\boldsymbol{x}, \xi)|}{A(\boldsymbol{x}, \boldsymbol{x}_s) \, A(\boldsymbol{x}, \boldsymbol{x}_r) \left| \nabla\tau(\boldsymbol{x}, \boldsymbol{x}_s) + \nabla\tau(\boldsymbol{x}, \boldsymbol{x}_r) \right| \left| H(\boldsymbol{x}, \boldsymbol{x}', \xi) \right|}$$

The integral (61) is exactly of the form of the integral (44), except that in the application here, the explicit dependence on the length scale, L, has been neglected. The condition that the wave number be large, in order to apply the results of the previous section, first requires that $\nabla[\tau(\boldsymbol{x}, \boldsymbol{x}_s) + \tau(\boldsymbol{x}, \boldsymbol{x}_r)]$ be nonzero. (Note that this gradient vanishes when the rays from $\boldsymbol{x}_s$ and $\boldsymbol{x}_r$ are tangent at $\boldsymbol{x}$. That is, they are both part of a single ray from $\boldsymbol{x}_s$ to $\boldsymbol{x}_r$. Such ray paths would naturally arise in transmission tomography. This theory does not apply to those applications. On the other hand, *diffraction tomography* falls in the range of applications of this theory.) That this sum of gradients does not vanish is assured by the assumption that $h(\boldsymbol{x}, \xi)$ be nonzero, because this vector is the first row of the determinant in (50). Once the minimum magnitude of this vector is determined, and the natural length scale L for the integral (53) is established, the burden of "large wave number" is put on the frequency parameter. In seismic applications, frequencies as low as 4 Hz will prove to be high enough for the corresponding parameter, $\lambda = KL$, to be large. Thus, the theory of the previous sections applies in seismic applications.

It follows from the analysis of the previous section that the output of (61) is proportional to a band limited singular function of the reflecting surface. The proportionality factor may be constructed from the discussion below (44) and the definition of f in (52). When $\boldsymbol{x}$ is on S, this factor reduces to $f(\boldsymbol{x}, \boldsymbol{x}, \hat{\boldsymbol{k}})$. However, the distinguished value of $\hat{\boldsymbol{k}}$ must be $\pm\hat{\boldsymbol{n}}$ as a consequence of applying stationarity to the derivatives in (14). (This condition is just Snell's law.) From the approximate form of $\hat{\boldsymbol{k}}$ in (58), it is apparent that $\hat{\boldsymbol{k}}$ must point *downward*, while the normal to S, $\hat{\boldsymbol{n}}$, in the Kirchhoff representation of u_S is an *upward* normal. Consequently, in (62)

$$\hat{\boldsymbol{n}} \cdot \nabla' \left[\tau(\boldsymbol{x}', \boldsymbol{x}_s) + \tau(\boldsymbol{x}', \boldsymbol{x}_r)\right] \Bigg|_{\boldsymbol{x}' = \boldsymbol{x}} = - \left|\nabla\tau(\boldsymbol{x}, \boldsymbol{x}_s) + \nabla\tau(\boldsymbol{x}, \boldsymbol{x}_r)\right| \ . \tag{63}$$

Furthermore, from (60),

$$H(\boldsymbol{x}, \boldsymbol{x}, \xi) = h(\boldsymbol{x}, \xi) \ . \tag{64}$$

Thus, when the distinguished point, $\boldsymbol{x}$ on S, is in the region where the neutralizer functions are equal to unity,

$$f(\boldsymbol{x}, \boldsymbol{x}, \hat{\boldsymbol{k}}) = R(\boldsymbol{x}, \boldsymbol{x}_s) \ . \tag{65}$$

In this equation, $\boldsymbol{x}_s$ is fixed by the stationarity conditions [(14) and the change of variables] such that Snell's law is satisfied. This is an angularly dependent reflection coefficient. It is a function of $\cos\Theta$, with Θ the supplement of the angle between $\hat{\boldsymbol{n}}$ and $\nabla\tau(\boldsymbol{x}, \boldsymbol{x}_s)$. At the stationary point, because Snell's law is satisfied, this is just half the angle between $\nabla\tau(\boldsymbol{x}, \boldsymbol{x}_s)$ and $\nabla\tau(\boldsymbol{x}, \boldsymbol{x}_r)$. The function, $\cos\Theta$ is determined in a manner similar to the way $\cos\theta$ and $\cos\phi$ were determined in the previous section, using the identity

$$\left|\nabla\tau(\boldsymbol{x}, \boldsymbol{x}_s) + \nabla\tau(\boldsymbol{x}, \boldsymbol{x}_r)\right| = 2\cos\Theta \,/\, c(\boldsymbol{x}) \ , \tag{66}$$

which can be confirmed by squaring both sides. See Bleistein [1987a] for details.

Application to Born-Approximate Data

In Beylkin [1985], the Born approximation was used to represent the upward scattered data. The inversion of such data for determination of medium parameters is more closely akin to the analysis of aperture-limited Fourier transforms of piecewise smooth functions. For such data, we already know from the previous sections what the nature of the aperture limited inversion of the data will be. For Beylkin's inversion operator, the output will be band limited step functions across the discontinuity surfaces of the integrand (usually, a perturbation in the index of refraction, or some equivalent). By making the modifications equivalent to changing the operator I to the operator I of the previous section, the output will again be singular functions of the discontinuity surfaces scaled by the jump in the integrand.

Thus, whether one applies the operator to Born data or Kirchhoff data, the result is essentially the same, a weighted singular function. The only issue is whether one interprets the weight in terms of the Born approximation or the Kirchhoff approximation. It is my view that the Kirchhoff approximation better represents the upward scattered field in most applications. One reason is that implicit in the large wave number asymptotics is the assumption of high frequency as seen above. The Kirchhoff approximation is a high frequency approximation. Furthermore, in many applications, the change in medium parameters across reflectors, or the angle of incidence and reflection with respect to the normal, may be large enough to call into question the use of the Born approximation.

Summary

It has been shown here that the process of applying a proposed inverse scattering operator to model data is asymptotically equivalent to aperture-limited Fourier inversion with a dual set of Fourier variables determined from the travel time functions from source to output point and receiver to output point. The inversion operator proposed here is a modification of an operator proposed by Beylkin [1985]. Where his inversion would lead to band limited step functions, this one leads to band limited singular functions. When the bandwidth is of sufficient extent, his operator leads to a Fourier approximation of the unknown function. When the bandwidth is not sufficient for this application, but can be characterized as "large wave number" in some sense, the inversion proposed here still provides a map of the discontinuity surface(s) (reflectors) of the unknown function, as well as a means of estimating reflection strength.

In Bleistein [1987a,b], these results are confirmed by applying asymptotic analysis directly to the integral (53). Here, it was shown by classical asymptotic methods that, in fact, this result is really a more fundamental phenomenon of Fourier inversion and the results for inverse scattering follow from those more fundamental results, once one identifies the Fourier variables.

For seismic applications, a reflector map would seem to be a desirable output of an inversion operator. However, Beylkin and his associates [in the above cited references] have certainly made a strong case for a more complete inversion when the data is available. The fact that the operator is deduced on the basis of a high-frequency assumption seems not to degrade the quality of the low frequency output to any significant degree.

CONCLUSIONS

It has been shown here by classical means that high frequency inversion of scattered data is equivalent to large wave number aperture-limited Fourier inversion. The type of output one obtains from the inversion operators of interest can be explained in terms of elementary ideas about such Fourier inversion. In particular, the transition from aperture-limited step function output to aperture-limited singular function output (across the steps) follows directly from the asymptotic analysis of large wave number aperture limited Fourier transforms.

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REFERENCES

Abramowitz, M., and Stegun, I. A., 1965, **Handbook of Mathematical Functions,** Dover, New York.

Beylkin, G., D. Miller, and M. Oristaglio, 1985, Spatial resolution of migration algorithms, in **Acoustic Imaging, 14,** Berkhout, A. J., Ridder, J., and van der Waal, L. F., Eds., Plenum Pub. Co., 155-167.

Bleistein, N., 1984, **Mathematical Methods for Wave Phenomena,** Academic Press, New York.

Bleistein, N., 1986, Two-and-one-half dimensional in- plane wave propagation:

Geophysical Prospecting, 44, 1034-1040.

Bleistein, N., 1987a, On the imaging of reflectors in the earth, **Geophysics, 52,** 931-942.

Bleistein, N., 1987b, Kirchhoff inversion for reflector imaging and soundspeed and density variations, **Deconvolution and Inversion,** Proceedings of a Joint EAEG/SEG Workshop, Rome, Italy, September, 1986, Blackwell Publishers, Oxford.

Bojarski, N. N., 1967, Three dimensional electromagnetic short pulse inverse scattering, Spec. Proj. Lab. Rep., Syracuse Univ. Res. Corp., Syracuse, New York.

Bojarski, N. N., 1968, Electromagnetic inverse scattering theory, Spec. Proj. Lab. Rep., Syracuse Univ. Res. Corp., Syracuse, New York.

Bojarski, N. N., 1982, A survey of the physicl optics inverse scattering identity, **IEEE Trans. Ant. and Prop., AP-30,** 980- 989.

Cohen, J. K., and Bleistein, N., 1979, The singular function of a surface and physical optics inverse scattering, **Wave Motion, 1,** 153-161.

Mager, R. D., and Bleistein, N., 1978, An examination of the limited aperture problem of physical optics inverse scattering, **IEEE Trans. Ant. and Prop., AP-17,** 308-314 (Correction, AP-18, 194.)

Miller, D., M. Oristaglio, and G. Beylkin, 1987, A new slant on seismic imaging: Migration and integral geometry: **Geophysics, 52,** 943-964.

Mora, P., 1987, Inversion equals migration plus tomagraphy, **Geophysics,** to appeaar.

Schneider, W. A., 1978, Integral formulation for migration in two and three dimensions, **Geophysics, 43,** 49-76.

Sullivan, M. F., and J. K. Cohen, 1985, Pre-stack Kirchhoff inversin of common offset data: **Geophysics, 52,** 745-754.

FIGURE CAPTIONS

Figure 1 The singular function of the line, $z=0$. Also, a reference "spike," to be used for amplitude comparisons.

Figure 2 The filter function applied in k-domain. The support of this function is the domain D_k.

Figure 3 The aperture limited inversion back to the spatial domain of the singular function data from Figure 1, restricted to the domain D_k of Figure 2. Reference spike in the corner for amplitude comparison.

Figure 4 An alternative filter function in the k-domain. D_k in this case does not include the direction (0,1) of the normal to the support of the singular function in Figure 1.

Figure 5 Inversion of aperture limited data for the singular function of Figure 1 using the domain D_k of Figure 4. Reference spike demonstrates that the scale is the same as in Figures 3 and 5. Inversion is effectively zero as predicted by the theory.

Figure 6 Same output as Figure 5, but scaled up by six orders of magnitude.

Figure 7 The singular function of a circle.

Figure 8 A filter containing all angles, limited magnitude of k. The domain D_k is an annulus.

Figure 9 Aperture limited version of the singular function of a circle using the domain D_k of Figure 8.

Figure 10 An alternative aperture to apply to the Fourier transform of the singular function of a circle.

Figure 11 Aperture limited inversion of Fourier data for the singular function of a circle. The circle is properly identified only for the region in which the normal to the circle lies in the angular aperture of D_k.

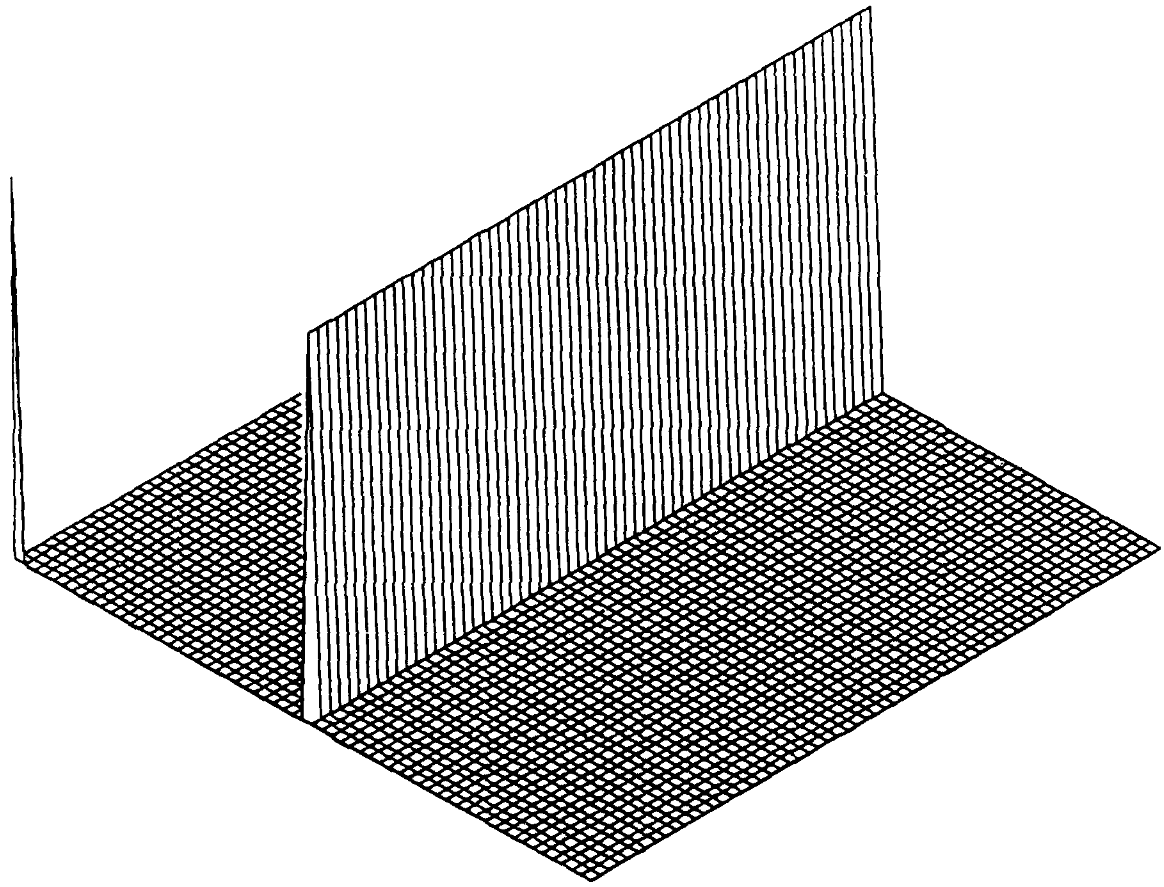

FIGURE 1

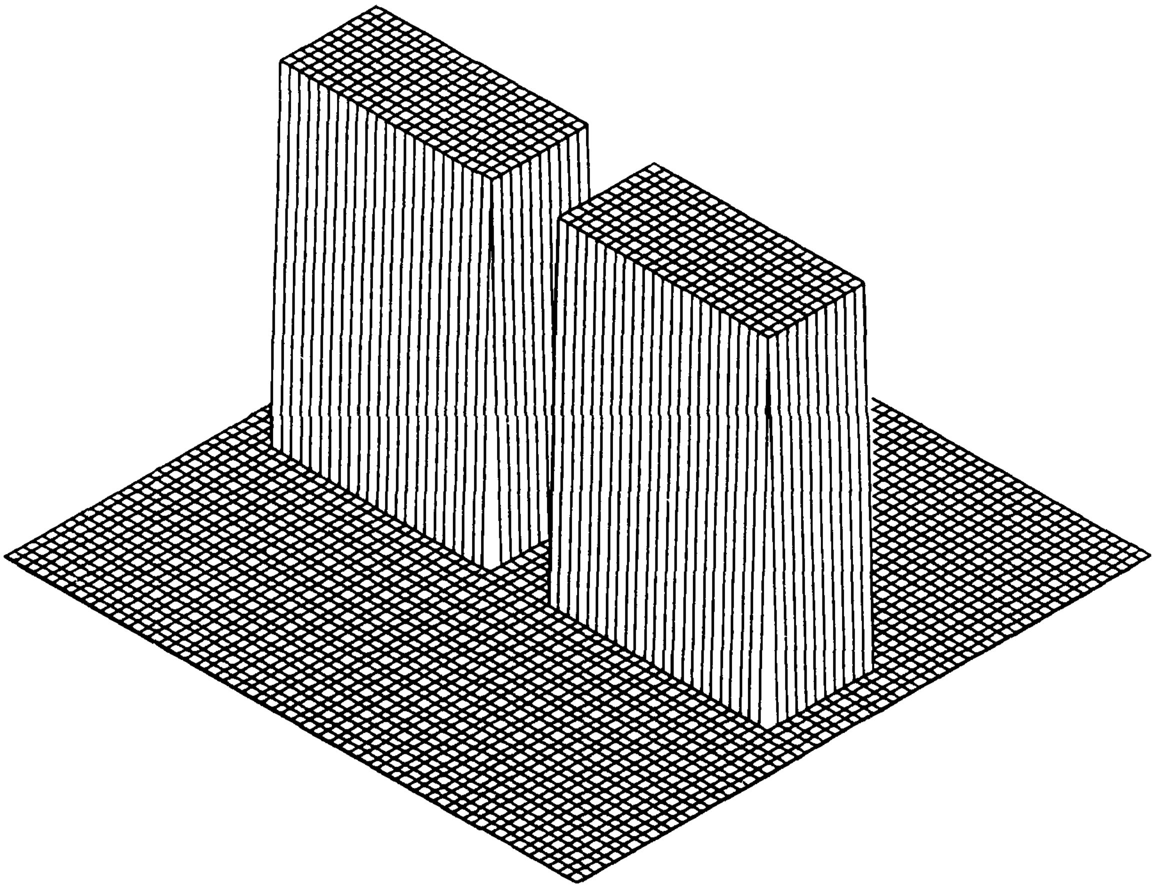

FIGURE 2

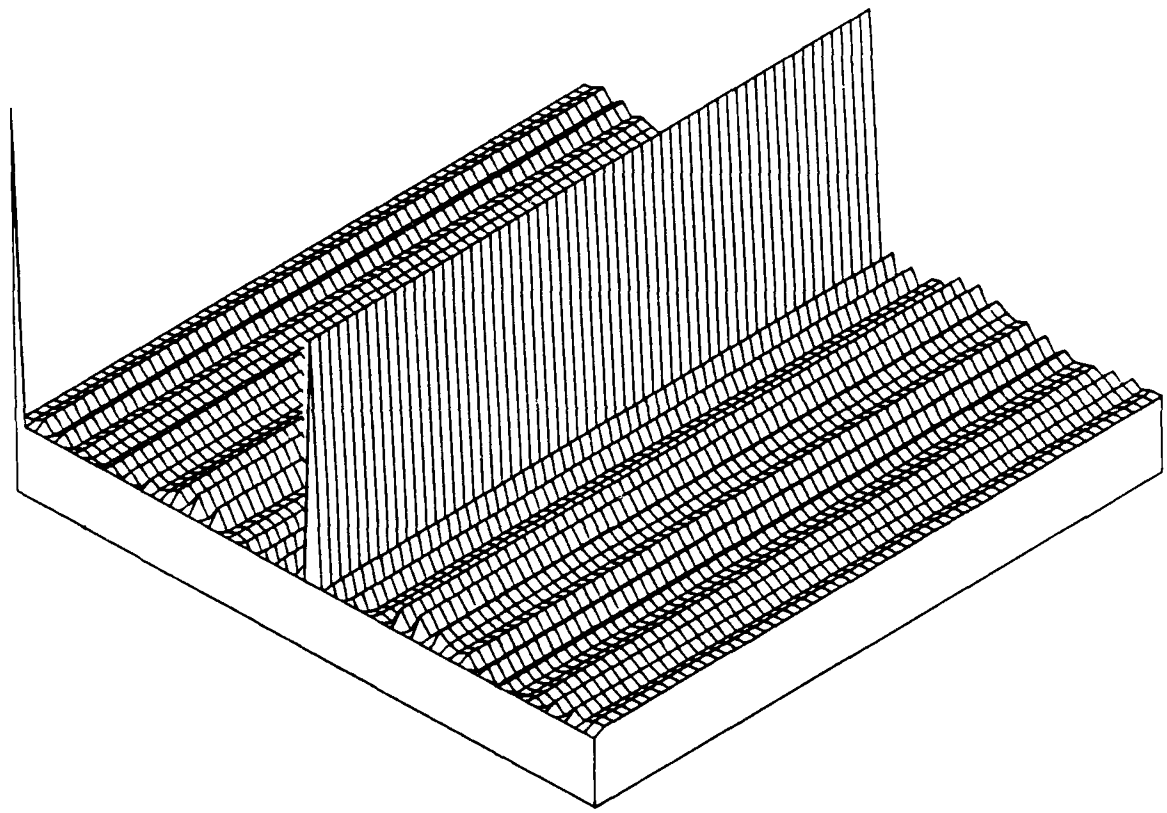

FIGURE 3

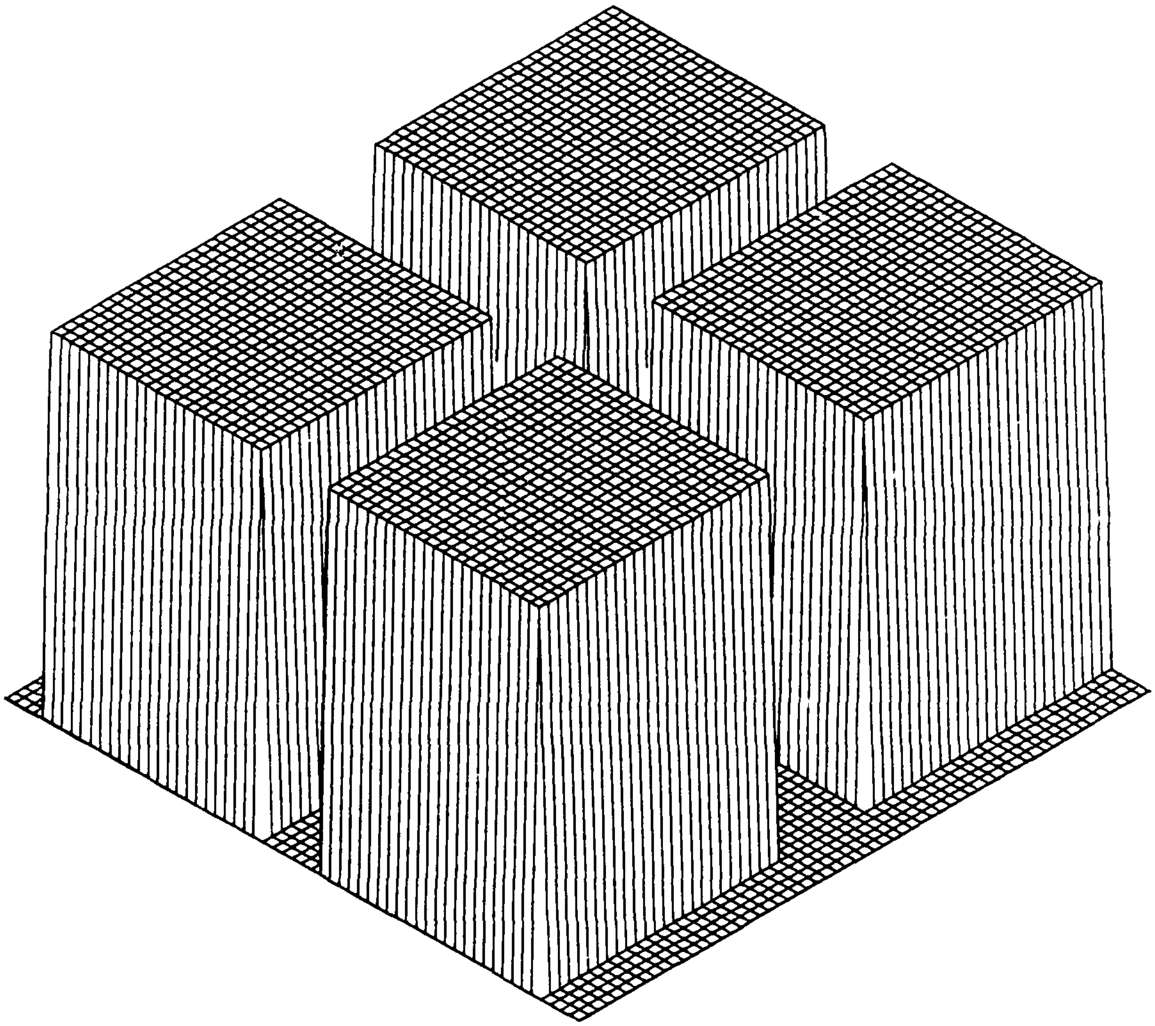

FIGURE 4

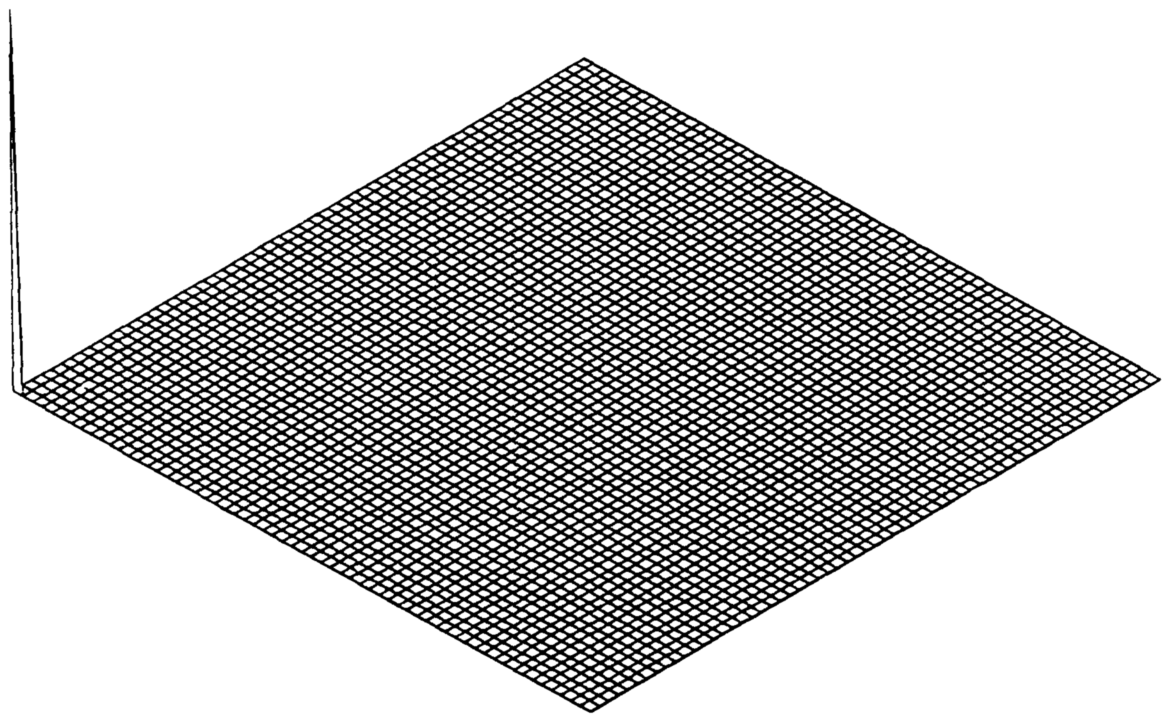

FIGURE 5

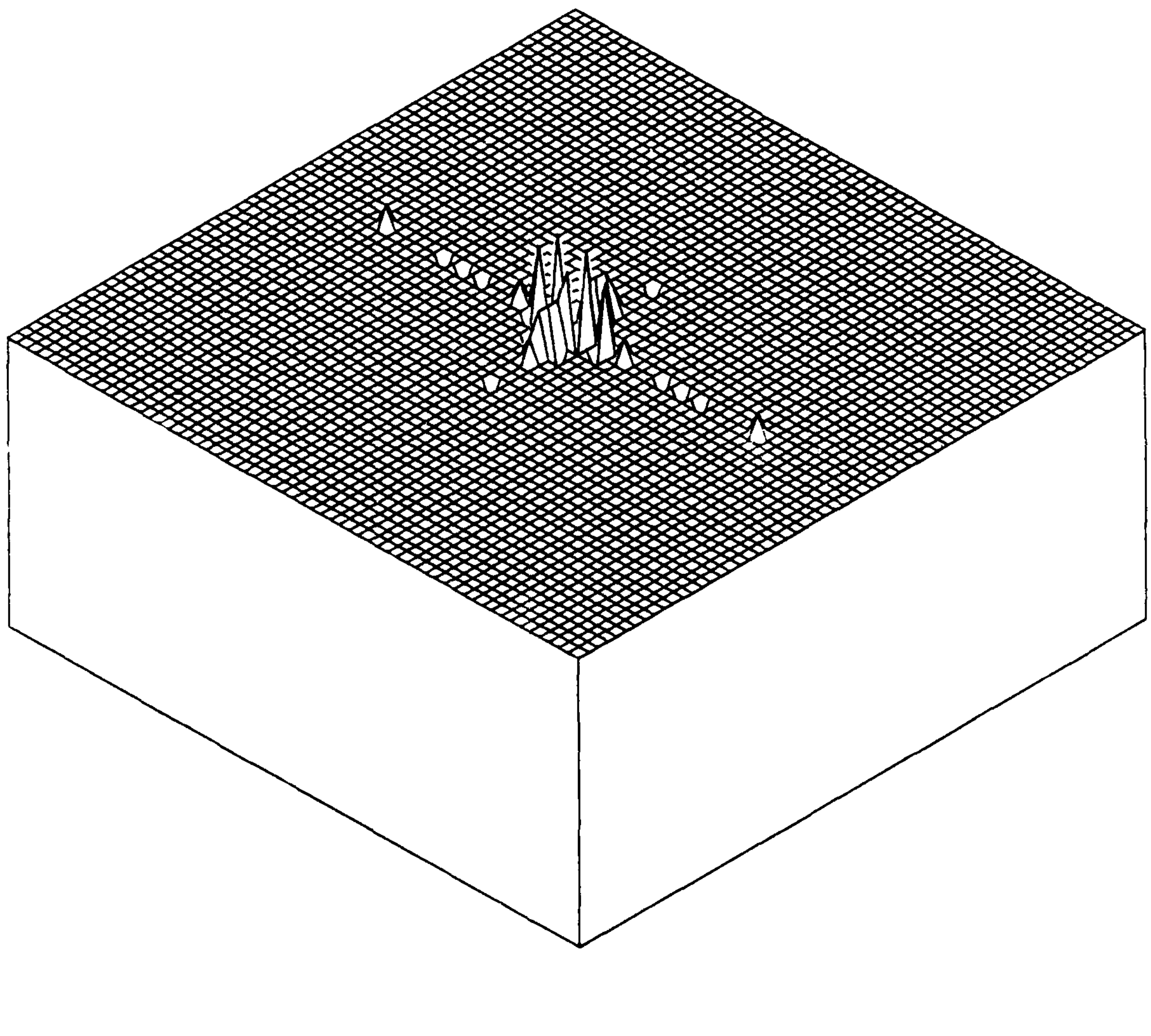

FIGURE 6

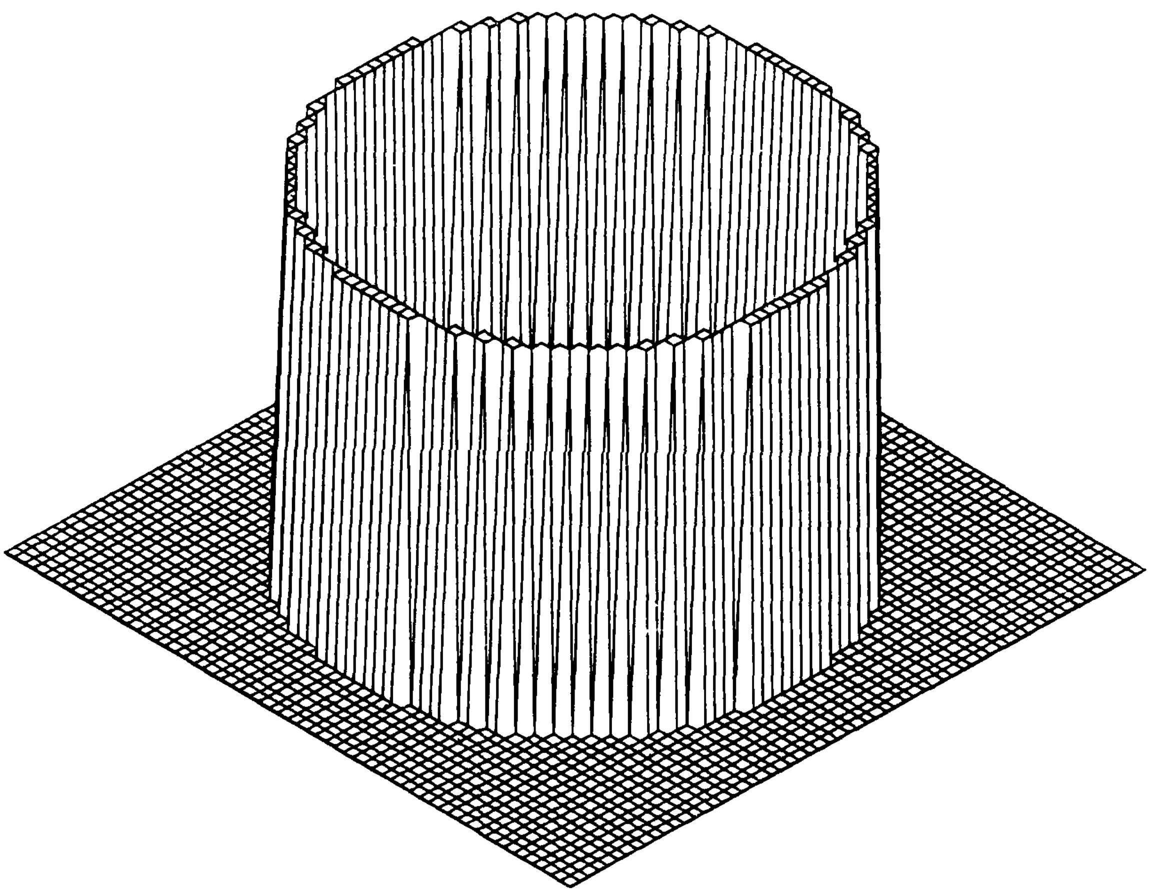

FIGURE 7

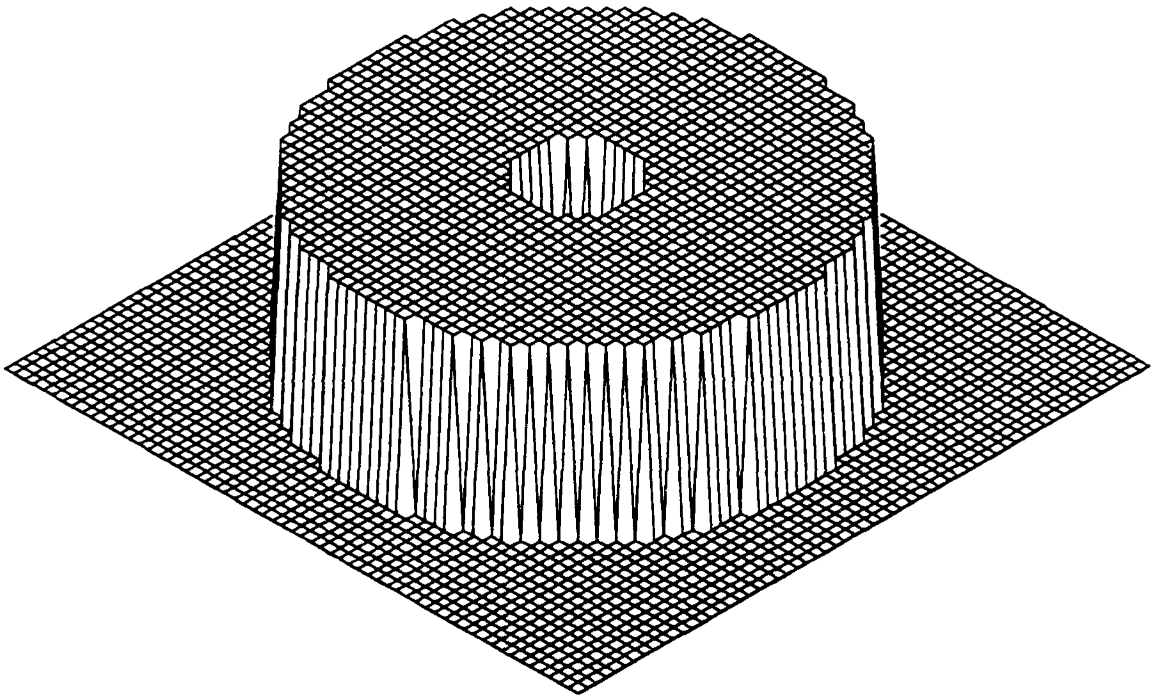

FIGURE 8

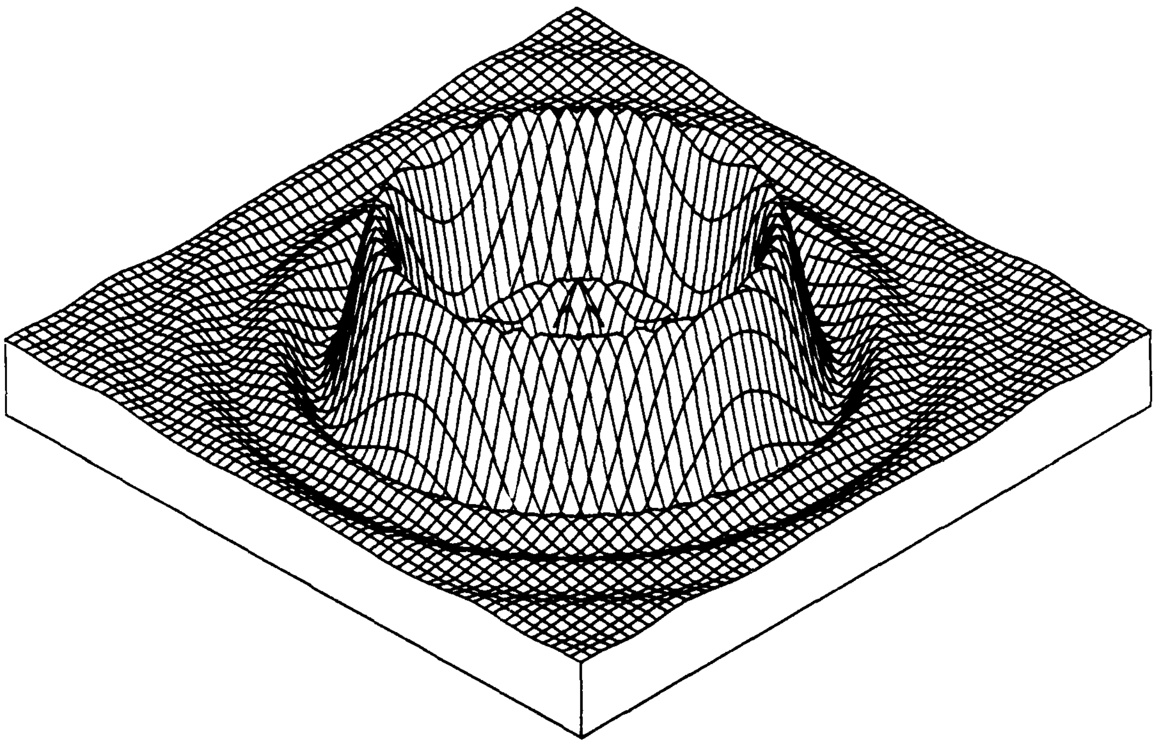

FIGURE 9

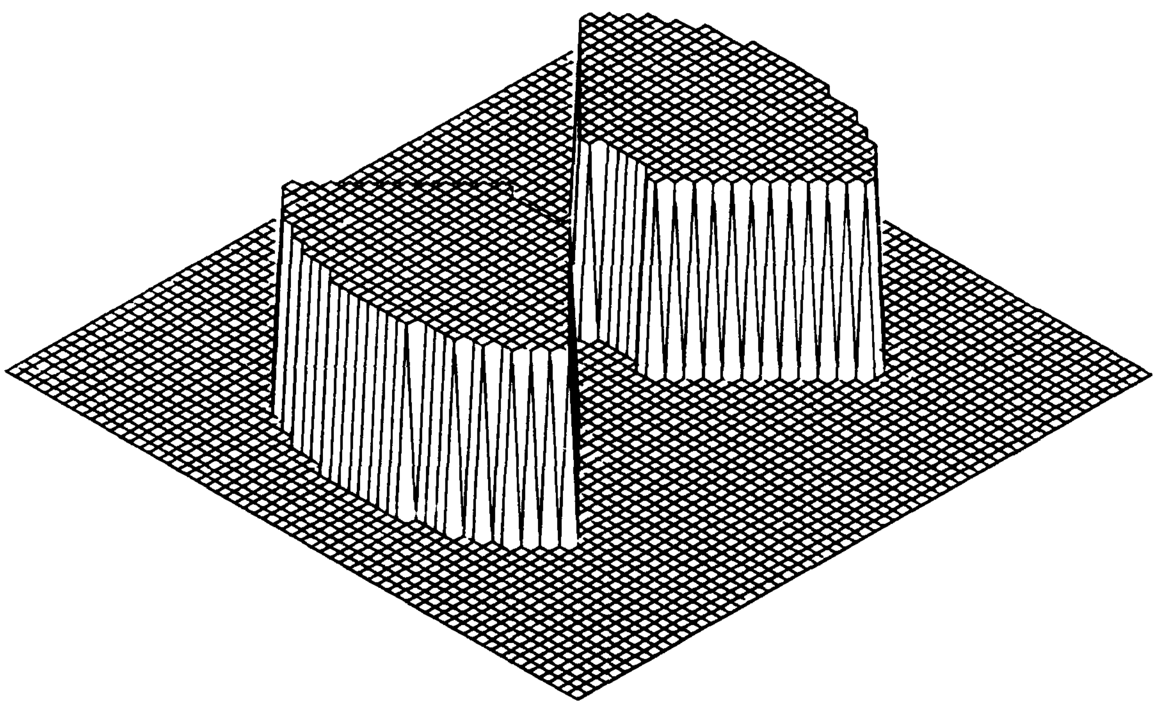

FIGURE 10

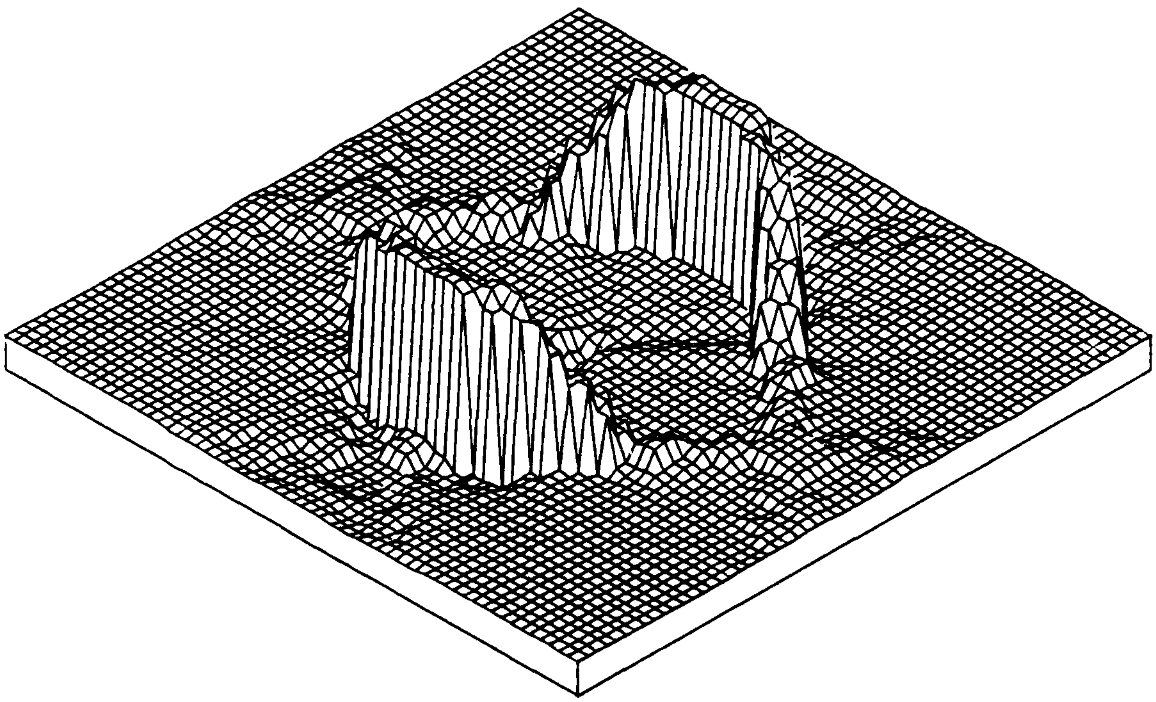

FIGURE 11

APPENDIX A

Determination of the determinant and signature of the matrix defined by (16) will be discussed here. The elements of the matrix are determined from (14) and (15). They are given by

$$\frac{\partial^2 \Phi}{\partial \sigma_i \, \partial \sigma_j} = -\tilde{\boldsymbol{p}} \cdot \frac{\partial^2 \boldsymbol{y}'}{\partial \sigma_i \, \partial \sigma_j} \; , \quad \frac{\partial^2 \Phi}{\partial \theta \, \partial \sigma_i} = -\tilde{\boldsymbol{\theta}} \cdot \frac{\partial \boldsymbol{y}'}{\partial \sigma_i} \quad ,$$

$$\frac{\partial^2 \Phi}{\partial \phi \, \partial \sigma_i} = -\sin\theta \, \tilde{\boldsymbol{\phi}} \cdot \frac{\partial \boldsymbol{y}'}{\partial \sigma_i} \; , \quad i, j = 1, 2$$

$$\frac{\partial^2 \Phi}{\partial \theta^2} = -\tilde{\boldsymbol{p}} \cdot (\boldsymbol{y} - \boldsymbol{y}') \; , \quad \frac{\partial^2 \Phi}{\partial \theta \, \partial \phi} = -\cos\theta \, \tilde{\boldsymbol{\phi}} \cdot (\boldsymbol{y} - \boldsymbol{y}')$$

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -\tilde{\boldsymbol{\rho}} \cdot (\boldsymbol{y} - \boldsymbol{y}') \; , \quad \tilde{\boldsymbol{\rho}} = (\cos\phi, \sin\phi) \quad . \tag{A1}$$

With no loss of generality, one can assume for this calculation that σ_1 and σ_2 are each arclength variables in the principal directions at the stationary point. In this case, the upper left two-by-two matrix of elements simplifies to

$$\frac{\partial^2 \Phi}{\partial \sigma_i \, \partial \sigma_j} = -\delta_{ij} \, \tilde{\boldsymbol{p}} \cdot \boldsymbol{\kappa}_j \; , \quad i, j = 1, 2 \quad . \tag{A2}$$

Here, δ_{ij} is the Kronecker delta, equal to unity for $i = j$, zero, otherwise. Also, in this case, the first derivatives with respect to σ_i in (A1) reduce to orthogonal unit vectors. In addition, the vectors $\tilde{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\phi}}$ are orthogonal unit vectors, co-planar with the unit tangents as a consequence of the conditions of stationarity. (Recall, the normal to $S_{y'}$ is colinear or anti-colinear with $\tilde{\boldsymbol{p}}$.) Therefore, to simplify notation below, introduce ψ as follows:

$$\tilde{\boldsymbol{\theta}} \cdot \frac{\partial \boldsymbol{y}'}{\partial \sigma_1} = \cos\psi \; , \quad \tilde{\boldsymbol{\phi}} \cdot \frac{\partial \boldsymbol{y}'}{\partial \sigma_1} = \sin\psi \quad ,$$

$$\tilde{\boldsymbol{\theta}} \cdot \frac{\partial \boldsymbol{y}'}{\partial \sigma_2} = -\sin\psi \; , \quad \tilde{\boldsymbol{\phi}} \cdot \frac{\partial \boldsymbol{y}'}{\partial \sigma_2} = \cos\psi \quad . \tag{A3}$$

Direct calculation now leads to the result,

$$\det\left[\Phi_{ij}\right] = \sin^2\theta\left[1-\mu_1\mu_3\kappa_1\,|\,\boldsymbol{y}-\boldsymbol{y}'\,|\,\right]$$

$$\cdot\left[1-\mu_2\mu_3\kappa_2\,|\,\boldsymbol{y}-\boldsymbol{y}'\,|\,\right] \quad . \tag{A4}$$

In this equation, the μ_j's are defined by equation (20) and the κ_j's are the principal curvatures as described in the text. This result can be seen to be independent of ψ. Thus, for the purpose of determining the signature of Φ_{ij}, ψ may be set equal to zero. The zeroes at $\theta=0$ or π are due to the singularities of the coordinate system at these points and can be disregarded here.

It can be seen from (A4) that the sgn Φ_{ij} can only change when $\boldsymbol{y}$ moves through a center of principal curvature of $S_{y'}$. Thus, for the purpose of determining sgn Φ_{ij} for $\boldsymbol{y}$ near $S_{y'}$, we can take $\boldsymbol{y}$ on $S_{y'}$; that is, we may set $|\,\boldsymbol{y}-\boldsymbol{y}'\,|=0$. In this case, we find that

$$\det\left[\Phi_{ij}-\lambda\delta_{ij}\right] = \sin^2\theta\left[\lambda^2+\lambda\mu_1\kappa_1-1\right]\left[\lambda^2+\lambda\mu_2\kappa_2-\sin^2\theta\right] \tag{A5}$$

Each factor here has two roots of opposite sign, leading to the conclusion that sgn $\Phi_{ij}=0$. This completes the proof.

APPENDIX B

Lemma 2 will be proven in this appendix. The fourfold integral $\beta_N^*(\boldsymbol{x})$ in $\boldsymbol{\xi}$ and $\boldsymbol{\sigma}$ is of the form

$$\beta_N^*(\boldsymbol{x}) = \int_{D_\eta} G(\boldsymbol{\eta})\exp\left[i\omega\,\Phi_1(\boldsymbol{\eta})\right]\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_4 \quad . \tag{B1}$$

In this equation, the four variables, $\boldsymbol{\eta}$, are just the variables, $(\xi_1,\xi_2,\sigma_1,\sigma_2)$, renamed to make the discussion below easier. The phase Φ_1 is just the phase Φ in the newly-named variables, while G identifies the full, neutralized amplitude of (54) multiplied by $N^*(\boldsymbol{x}',\boldsymbol{x})$. The domain, D_η, is just the pair of domains in $\boldsymbol{\xi}$ and $\boldsymbol{\sigma}$: $S_\xi\times S_\sigma$. The dependence of the integrand on $\boldsymbol{x}$ has been surpressed because it is not important in this discussion.

The integrand has no stationary points in D_η and vanishes C^∞ smoothly on its boundary. Therefore, the integrand is expanded as was done in the previous section [eq. (9)], in preparation for integration by parts:

$$G(\boldsymbol{\eta})\exp\left[i\omega\,\Phi_1(\boldsymbol{\eta})\right] = \frac{1}{i\omega}\left[\nabla_\eta\cdot\left[\frac{\nabla\Phi_1}{\left|\nabla\Phi_1\right|^2}\,G(\boldsymbol{\eta})\,\exp\left[i\lambda\Phi_1\right]\right] + G_1(\boldsymbol{\eta})\,\exp\left[i\lambda\Phi_1\right]\right] \;,$$

$$G_1(\boldsymbol{\eta}) = -\nabla_n\cdot\left[\frac{\nabla\Phi_1}{\left|\nabla\Phi_1\right|^2}\,G(\boldsymbol{\eta})\right] \;. \tag{B2}$$

Substitute this identity into the previous equation and use the divergence theorem to replace the first integral over D_η by an integral over the boundary of this domain. Since the integrand vanishes on the boundary, this first integral is zero and

$$\beta_N^*(\boldsymbol{x}) = \frac{1}{i\omega}\int_{D_\eta} G_1(\boldsymbol{\eta})\exp\left[i\omega\,\Phi_1(\boldsymbol{\eta})\right]\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_4 \;. \tag{B3}$$

This process can be repeated recursively, as long as the integrand has derivatives in D_η. Each time, another power of $1/i\omega$ is introduced as a multiplier of the integrand while the integral, itself, remains finite. Thus, the claim of the lemma is confirmed.

APPENDIX C

Lemma 3 will be proven here. For brevity, set

$$\Psi(\boldsymbol{x}, \boldsymbol{x}_s, \boldsymbol{x}_r) = \tau(\boldsymbol{x}, \boldsymbol{x}_s) + \tau(\boldsymbol{x}, \boldsymbol{x}_r) \;, \tag{C1}$$

and write the Taylor series for Ψ as

$$\Psi(\boldsymbol{x}', \boldsymbol{x}_s, \boldsymbol{x}_r) = \nabla\Psi(\boldsymbol{x}', \boldsymbol{x}_s, \boldsymbol{x}_r) + \sum_{n=2}^{\infty}\sum_{\|\boldsymbol{\nu}\|=n}\frac{1}{\boldsymbol{\nu}!}\,\frac{\partial^n\Psi(\boldsymbol{x}', \boldsymbol{x}_s, \boldsymbol{x}_r)}{\partial\boldsymbol{x}^{\boldsymbol{\nu}}}\,(\boldsymbol{x}'-\boldsymbol{x})^{\boldsymbol{\nu}} \;. \tag{C2}$$

In this equation,

$$\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3) \;, \quad \|\boldsymbol{\nu}\| = \nu_1 + \nu_2 + \nu_3 \;,$$

$$\boldsymbol{\nu}! = \nu_1!\,\nu_2!\,\nu_3!\ , \quad (\boldsymbol{x}' - \boldsymbol{x})^{\boldsymbol{\nu}} = (x_1' - x_1)^{\nu_1}\,(x_2' - x_2)^{\nu_2}\,(x_3' - x_3)^{\nu_3}$$

$$\partial \boldsymbol{x}^{\boldsymbol{\nu}} = \partial x_1^{\nu_1}\,\partial x_2^{\nu_2}\,\partial x_3^{\nu_3}\ , \tag{C3}$$

with the ν_j's being nonnegative integers.

Now define the vector $\boldsymbol{k}$ by

$$\boldsymbol{k}_j = \omega \boldsymbol{p}_j\ ,$$

$$p_j = \frac{\partial \Psi}{\partial x_j} + \sum_{n=2}^{\infty} \sum_{\|\boldsymbol{\nu}\| = n} \frac{\nu_j}{\|\boldsymbol{\nu}\|} \frac{1}{\boldsymbol{\nu}!} \frac{\partial^n \Psi}{\partial \boldsymbol{x}^{\boldsymbol{\nu}}} \tag{C4}$$

(C4)

One can check that $\boldsymbol{k}\cdot(\boldsymbol{x}' - \boldsymbol{x}) eq \omega\Psi$ by multiplying p_j by $\omega(x_j' - x_j$ and summing on j to obtain the series in (C2) multiplied by ω. Note that the radius of convergence of each of the series in (C4) is the same as for the series in (C2). Furthermore, (59) follows by direct computation. Thus, in some neighborhood of $\boldsymbol{x}$, the transformation from (ω, ξ) to $\boldsymbol{x}$ is one-to-one. Indeed, with hindsight, choose $N(\boldsymbol{x}', \boldsymbol{x})$ to be small enough that $H(\boldsymbol{x}, \xi)$ is nonvanishing on the support of $N(\boldsymbol{x}', \boldsymbol{x})$.

This completes the proof.